Math 115 — Practice for Exam 3

Generated March 16, 2017

Name: SOLUTIONS

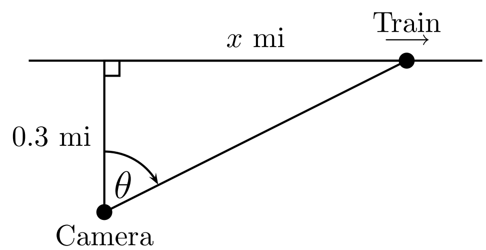
Instructor: ______________________ Section Number: __________

1. This exam has 9 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	**Exam**	**Problem**	**Name**	**Points**	**Score**
Winter 2010	3	8	train+camera	12	
Fall 2011	3	9	white whale	10	
Winter 2012	3	1	ship	8	
Fall 2012	3	9	ice cube	11	
Winter 2014	3	6	gold platinum	8	
Winter 2015	3	4	chocolate cone	12	
Fall 2015	3	5	baseball	9	
Winter 2016	3	3	shadow	8	
Fall 2016	3	2	aluminum cone	9	
Total				87	

Recommended time (based on points): 105 minutes

8. [12 points]
A train is traveling eastward at a speed of 0.4 miles per minute along a long straight track, and a video camera is stationed 0.3 miles from the track, as shown in the figure. The camera stays in place, but it rotates to focus on the train as it moves.
Suppose that t is the number of minutes that have passed since the train was directly north of the camera; after t minutes, the train has moved x miles to the east, and the camera has rotated θ radians from its original position.



a. [3 points] Write an equation that expresses the relationship between x and θ.

Solution:

$$\tan(\theta) = \frac{x}{0.3}, \text{ or } \theta = \arctan\left(\frac{x}{0.3}\right)$$

b. [4 points] Suppose that seven minutes have passed since the train was directly north of the camera. How far has the train moved in this time, and how much has the camera rotated?

Solution: The train's velocity is constant (i.e. $\frac{dx}{dt} = 0.4$), so we can use the formula *distance* $=$ *velocity* $\cdot$ *time*. Thus, the train has moved $\left(0.4\frac{\text{mi}}{\text{min}}\right)(7\text{min}) = 2.8$ miles.

Using the fact that $x = 2.8$, we have $\theta = \tan^{-1}(\frac{2.8}{0.3}) \approx 1.4641$, so the camera has rotated 1.4641 radians in the clockwise direction.

c. [5 points] How fast is the camera rotating (in radians per minute) when $t = 7$?

Solution: We want to find $\frac{d\theta}{dt}$. To do so, we will (implicitly) take the derivative of our equation from part (a), with respect to t.

$$\frac{d}{dt}\tan(\theta) = \frac{d}{dt}\left(\frac{x}{0.3}\right), \text{ or } \frac{1}{\cos^2(\theta)}\frac{d\theta}{dt} = \frac{1}{0.3}\frac{dx}{dt}$$

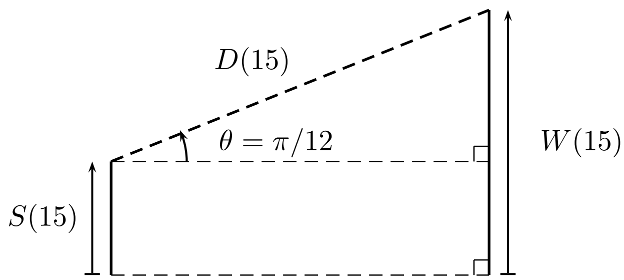
We can then plug in $\frac{dx}{dt} = 0.4$ and $\theta = 1.4641$:

$$\frac{1}{\cos^2(1.4641)}\frac{d\theta}{dt} = \frac{0.4}{0.3}, \text{ so } \frac{d\theta}{dt} = \frac{0.4}{0.3}\cos^2(1.4641) \approx 0.01513.$$

Thus, the camera is rotating at a speed of 0.01513 radians per minute (in the clockwise direction) when $t = 7$.

9. [10 points] You are sitting on a ship traveling at a constant speed of 6 ft/sec, and you spot a white object due east moving parallel to the ship. After tracking the object through your telescope for 15 seconds, you identify it as the white whale. Let $W(t)$ denote the distance of the whale from its starting point in feet, and $S(t)$ denote the distance of the ship from its starting point in feet, with t the time in seconds since you saw the whale. You also note that at the end of the 15 seconds you were tracking the whale, you have turned your head $\pi/12$ radians north to keep it in your sights.

a. [1 point] If initially the creature is 5280 ft (1 mile) from the ship due east, use the angle you have turned your head to find the distance $D(t)$ in feet between the ship and the creature at 15 seconds. The figure below should help you visualize the situation. Recall that, for right triangles, $\cos(\theta)$ is the ratio of the adjacent side to the hypotenuse.



Solution: Since $\cos(\pi/12) = \frac{5280}{D(15)}$, we find that $D(15) = \frac{5280}{\cos(\pi/12)} \approx 5466.258$ft.

b. [2 points] Let $\theta(t)$ give the angle you've turned your head after t seconds of tracking the whale. Write an equation $D(t)$ for the distance between the ship and the whale at time t (Hint: your answer may involve $\theta(t)$).

Solution: From the previous part, we know that the distance between the ship and the whale is 5280 divided by the cosine of the angle you've turned your head. Since $\theta(t)$ gives how far you've turned your head, we can find the distance at any time t using the function $D(t) = \frac{5280}{\cos(\theta(t))}$.

c. [3 points] If at precisely 15 seconds, you are turning your head at a rate of .01 radians per second, what is the instantaneous rate of change of the distance between the ship and the whale?

Solution: Since $D(t) = \frac{5280}{\cos(\theta(t))}$, we take the derivative with respect to t on both sides to get

$$\frac{dD}{dt} = \frac{5280\sin(\theta(t))}{\cos^2(\theta(t))}\theta'(t).$$

Since $\theta(15) = \pi/12$ and we are given $\theta'(15) = .01$, we get

$$\left.\frac{dD}{dt}\right|_{t=15} = \frac{5280\sin(\pi/12)}{\cos^2(\pi/12)}(.01) \approx 14.6468\text{ft/sec}.$$

d. [4 points] What is the speed of the whale at $t = 15$ seconds? Hint: Use the Pythagorean theorem.

Solution: The right triangle in the figure above has hypotenuse $D(t)$ and sides with length $D(t)$ and $W(t) - S(t)$, so the Pythagorean theorem states

$$D(t)^2 = 5280^2 + (W(t) - S(t))^2.$$

If we take the t derivative of both sides, we get

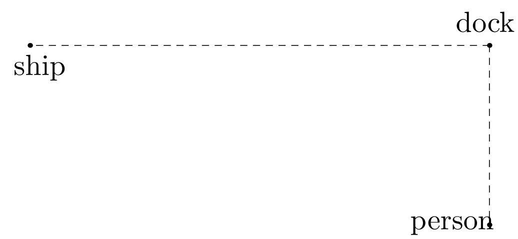
$$2D(t)\frac{dD}{dt} = 2(W(t) - S(t))\left(\frac{dW}{dt} - \frac{dS}{dt}\right).$$

To find $\left.\frac{dW}{dt}\right|_{t=15}$, we will need $D(15) = 5466.258$ and $\left.\frac{dD}{dt}\right|_{t=15} = 14.6468$. We also need to find $W(15) - S(15)$, but since this is one side of the right triangle, we can use tangent to find this distance: $W(15) - S(15) = 5280\tan(\pi/12) \approx 1414.7717$. Finally, we also need to know $\frac{dS}{dt}$, but in the description of the problem it says that ship is traveling at a constant speed of 6 ft/sec. Plugging all of this information into our equation we have

$$2(5466.258)(14.6468) = 2(1414.7717)\left(\frac{dW}{dt} - 6\right) \Rightarrow 56.5909 = \left.\frac{dW}{dt}\right|_{t=15} - 6.$$

Therefore, $\left.\frac{dW}{dt}\right|_{t=15} \approx 62.5909$ ft/sec

1. [8 points] A ship is sailing out to sea from a dock, moving in a straight line perpendicular to the coast. At the same time, a person is running along the coast toward the dock, hoping desperately to jump aboard the departing ship. Let $b(t)$ denote the distance in feet between the ship and the dock t seconds after its departure, and let $p(t)$ denote the distance in feet between the person and the dock t seconds after the ship's departure. The situation is depicted below for your reference:



Suppose that 10 seconds after the ship's departure, it is 40 feet from the dock and is sailing away at a speed of 20 ft/sec. At the same moment, the person is 30 feet from the dock and running toward it at 14 ft/sec.

a. [2 points] What is $b'(10)$? What is $p'(10)$?

Solution: Since the distance between the ship and the dock is increasing at 20 ft/sec while the distance between the person and the dock is decreasing at 14 ft/sec, we have

$$b'(10) = 20, \text{ and } p'(10) = -14.$$

b. [6 points] Is the distance between the person and the ship increasing or decreasing 10 seconds after the ship's departure? How fast is it increasing or decreasing? (Include units in your answer, and keep in mind that distance is measured along a straight line joining the person and the ship.)

Solution: If the distance between the ship and the person is denoted t seconds after the ship's departure is denoted $d(t)$, then by the Pythagorean Theorem,

$$b(t)^2 + p(t)^2 = d(t)^2.$$

Differentiating this gives:

$$2b(t)b'(t) + 2p(t)p'(t) = 2d(t)d'(t).$$

We know that $b(10) = 40$, $p(10) = 30$, and $d(10) = \sqrt{40^2 + 30^2} = 50$. If we plug this information as well as the numbers from part (a) into the above equation and solve for $d'(t)$, we find

$$d'(t) = \frac{760}{100} = 7.6.$$

Since this is positive, the distance is increasing at 7.6 ft/sec.

9. [11 points] A cube of ice is removed from the freezer and begins to melt. Let $\ell(t)$ be its side length, $V(t)$ its volume, and $S(t)$ its surface area, all dependent upon t, the number of minutes since it was removed from the freezer. The ice cube is melting (its volume is changing) at a rate proportional to its surface area. That is, $\frac{dV}{dt} = kS(t)$, for some number k. Initially the ice cube has a side length of 2 inches.

a. [4 points] Write V and S in terms of ℓ. Calculate the rate of change (with respect to time) of the side length of the ice cube in terms of ℓ and k.

Solution: $V = \ell^3$ and $S = 6\ell^2$.
We know, by the chain rule, that

$$\frac{dV}{dt} = \frac{dV}{d\ell}\frac{d\ell}{dt} = 3\ell^2\frac{d\ell}{dt}.$$

We are given that

$$\frac{dV}{dt} = kS(t) = k6\ell^2.$$

Setting these equal, we get

$$3\ell^2\frac{d\ell}{dt} = k6\ell^2.$$

So $\frac{d\ell}{dt} = 2k$ inches per minute.

b. [2 points] How fast is the **volume** of the ice cube changing immediately after it is removed from the freezer? Your answer will involve k.

Solution: The side length at time $t = 0$ is 2 inches, so
$\frac{dV}{dt}\Big|_{t=0} = k \cdot 6 \cdot 2^2 = 24k$ in^3 per minute

c. [2 points] What is the sign of k? Briefly explain.

Solution: From part (b) we see that k has the same sign as $\frac{dV}{dt}\Big|_{t=0}$, which is negative, since the volume is decreasing. Thus, k must be negative as well.

d. [3 points] How long will it take the ice cube to melt completely? Your answer may involve k.

Solution: The side length is changing at a constant rate of $2k$ and starts at 2 inches, so it will take T minutes, where $2 + 2kT = 0$. So $T = -\frac{1}{k}$ minutes.

6. [8 points] Suppose that a ring made entirely of gold and platinum is made from g ounces of gold and p ounces of platinum and that gold costs h dollars per ounce and platinum costs k dollars per ounce. Then the value, in dollars, of the ring is given by

$$v = gh + pk.$$

a. [3 points] Pat has a ring made entirely of gold and platinum. Pat's ring is made from 0.25 ounces of gold and 0.15 ounces of platinum. Suppose that the cost of gold is decreasing at an instantaneous rate of \$20 per ounce per year while the cost of platinum is increasing at an instantaneous rate of \$30 per ounce per year. At what instantaneous rate is the value of Pat's ring increasing or decreasing? *Remember to include units in your answer.*

Solution: In this setting, g is constant at 0.25 and p is constant at 0.15, and both h and k are changing. Differentiating with respect to t, we have

$$\frac{dv}{dt} = 0.25\frac{dh}{dt} + 0.15\frac{dk}{dt}.$$

Plugging in $\frac{dh}{dt} = -20$ and $\frac{dk}{dt} = 30$ yields $\frac{dv}{dt} = -5 + 4.5 = -0.5$.

Answer: The value of Pat's ring is (circle one) INCREASING **[DECREASING]**

at a rate of \$0.50 **per ounce per year**

b. [5 points] Jordan wants to design a ring made entirely of gold and platinum with a current value of \$900. Currently, gold costs \$1200 per ounce and platinum costs \$1500 per ounce. Let $w(p)$ be the total weight of Jordan's ring, in ounces, if p ounces of platinum are used.

(i) In the context of this problem, what is the domain of $w(p)$?

Solution: If the ring is all gold, then we use 0 ounces of platinum. Since the ring is worth \$900, the most platinum we could possibly use is $900/1500 = 0.6$ ounces.

Answer: $[0, 0.6]$

(ii) Find a formula for $w(p)$. No variables other than p should appear in your answer.

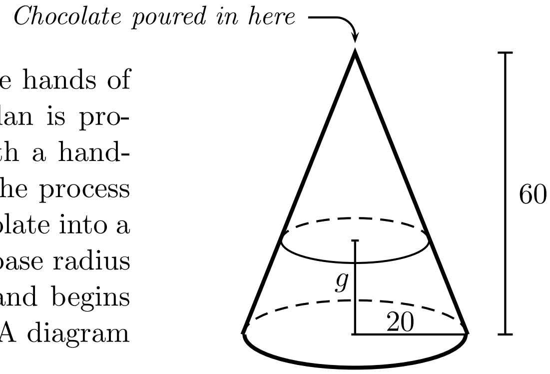
Solution: Since the value must be \$900, we have $900 = 1200g + 1500p$, or $g = 0.75 - 1.25p$. The total weight is therefore $w(p) = g + p = 0.75 - 0.25p$.

Answer: $w(p) =$ $0.75 - 0.25p$

(iii) How much gold and platinum should be in the ring if Jordan wants to minimize the weight of the ring? *You do not need to justify your answer.*

Solution: Since $w(p)$ is linear with negative slope, the smallest value will occur when p is greatest. Therefore, it occurs at $p = 0.6$, the right endpoint of our domain, where we use 0.6 ounces of platinum and 0 ounces of gold.

Answer: 0 ounces of gold and 0.6 ounces of platinum



4. [12 points]
Having taken care of Sebastian and sent Erin into the hands of the *Illumisqati*, King Roderick is pleased that his plan is proceeding well. Our wicked villain decides to relax with a handmade chocolate before he heads to his farmhouse. The process of making the chocolate involves pouring molten chocolate into a mould. The mould is a cone with height 60 mm and base radius 20 mm. Roderick places the mould on the ground and begins pouring the chocolate through the apex of the cone. A diagram of the situation is shown on the right.

In case they are helpful, recall the following formulas for a cone of radius r and height h:

$$\text{Volume} = \tfrac{1}{3}\pi r^2 h \quad \text{and} \quad \text{Surface Area} = \pi r(r + \sqrt{h^2 + r^2}).$$

a. [6 points] Let g be the depth of the chocolate, in mm, as shown in the diagram above. What is the value of g when Roderick has poured a total of 20,000 mm^3 of chocolate into the mould? *Show your work carefully, and make sure your answer is accurate to at least two decimal places.*

Solution: The volume of the solid is given by $V = \frac{1}{3}\pi(20)^2 60 - \frac{1}{3}\pi r^2(60-g)$ where r is the radius of the cross-section at height g. We want to rewrite r in terms of g. Using similar triangles we find the equation
$$\frac{r}{20} = \frac{60-g}{60},$$
which implies $r = \frac{60-g}{3}$. Therefore, $V = 8000\pi - \frac{\pi}{27}(60-g)^3$. So, to find the appropriate g we need to solve $8000\pi - \frac{\pi}{27}(60-g)^3 =$20,000. Solving, we get
$$(60-g)^3 = \frac{27}{\pi}(8000\pi - 20,000), \qquad (2)$$
which implies $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)} \approx 24.67$. The chocolate is approximately 24.67 mm deep when he has poured a total of 20,000 mm^3 of chocolate into the mould.

Answer: $g \approx$ **24.67**

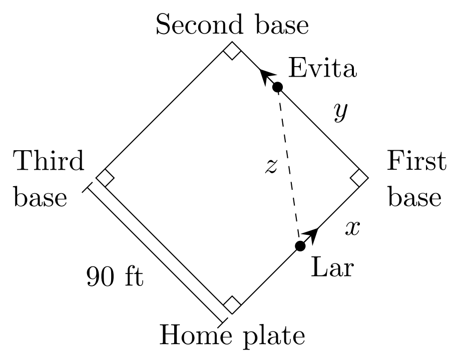
b. [6 points] How fast is the depth of the chocolate in the mould (g in the diagram above) changing when Roderick has already poured 20,000 mm^3 of chocolate into the mould if he is pouring at a rate of 5,000 mm^3 per second at this time? *Show your work carefully and make sure your answer is accurate to at least two decimal places. Be sure to include units.*

Solution: We want to find $\frac{dg}{dt}$ when $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)}$ (from part (a)) if $\frac{dV}{dt} = 5000$ at this time. Differentiating our formula (2) from part (a) with respect to t, we have
$$\frac{dV}{dt} = \frac{\pi}{9}(60-g)^2\frac{dg}{dt}.$$
Substituting $g = 60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)}$ and $\frac{dV}{dt} = 5000$ into this equation, we find
$$5000 = \frac{\pi}{9}\left(60 - \left[60 - \sqrt[3]{\frac{27}{\pi}(8000\pi - 20,000)}\right]\right)^2 \frac{dg}{dt} \quad \text{so}$$
$$\frac{dg}{dt} = \frac{5000 \cdot 9}{\pi}\left(\frac{27}{\pi}(8000\pi - 20,000)\right)^{-2/3} \approx 11.47.$$
(Using our approximation $g \approx 24.67$ instead gives us $\frac{dg}{dt} \approx 11.48$.)
So the depth of the chocolate is increasing at an instantaneous rate of about 11.47 mm/sec at that moment.

Answer: **11.47 mm/ sec.**

5. [9 points]
During the annual Srebmun Foyoj kickball game, Lar Getni kicks the ball and runs from home plate to first base, while Evita Vired runs from first base to second base.

Let x be the distance between Lar and first base, y be the distance between Evita and first base, and z be the distance between Lar and Evita, as shown in the diagram on the right. Note that the bases are arranged in a square and that the distance between consecutive bases is 90 feet.



At the moment when Lar is halfway from home plate to first base, Evita is two thirds of the way from first base to second base. At this moment, Lar is running at a speed of 32 ft/s, and Evita is running at a speed of 36 ft/s. The questions below all refer to this moment.

Throughout this problem, remember to show your work clearly, and include units in your answers.

a. [5 points] At the moment when Lar is halfway to first base, at what rate is the distance between Lar and Evita changing? Is the distance increasing or decreasing?

Solution: We need to find $\frac{dz}{dt}$ at the moment when Lar is halfway to first base. By the Pythagorean Theorem, we have $z^2 = x^2 + y^2$. Differentiating both sides of this equation with respect to time, we find that $2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$.

At the moment in question, we have $x = 45$ ft, $y = 60$ ft, $z = \sqrt{45^2 + 60^2} = 75$ ft, $\frac{dx}{dt} = -32$ ft/s, and $\frac{dy}{dt} = 36$ ft/s. So at this moment,

$$\frac{dz}{dt} = \frac{2(45)(-32) + 2(60)(36)}{2(75)} = 9.6 \text{ ft/s}.$$

Answer: The distance is (circle one) [INCREASING] DECREASING

at a rate of 9.6 **ft/s** .

b. [4 points] At the moment when Lar is halfway to first base, at what rate is the area of the right triangle formed by Lar, Evita, and first base changing? Is the area increasing or decreasing?

Solution: The area of the triangle is given by $A = \frac{xy}{2}$. Differentiating both sides of this equation with respect to time, we find that $\frac{dA}{dt} = \frac{x\frac{dy}{dt} + y\frac{dx}{dt}}{2}$.

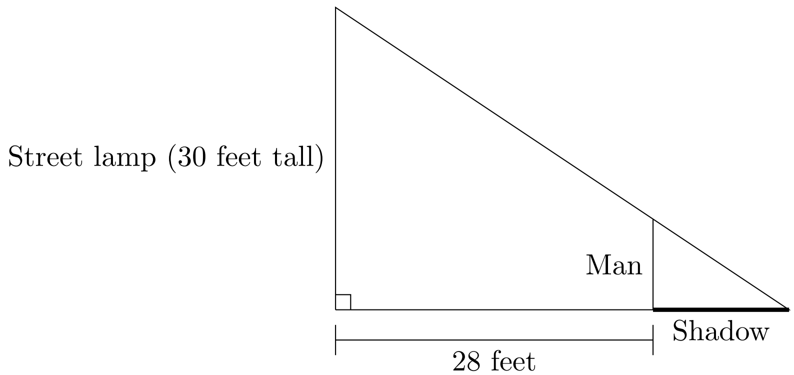
At the moment when Lar is halfway to first base, we therefore have

$$\frac{dA}{dt} = \frac{(45)(36) + (60)(-32)}{2} = -150 \text{ ft}^2\text{/s}.$$

Answer: The area is (circle one) INCREASING [DECREASING]

at a rate of 150 $\mathbf{ft^2/s}$.

3. [8 points] A man, who is 28 feet away from a 30 foot tall street lamp, is sinking into quicksand. (*See diagram below.*) At the moment when 6 feet of him are above the ground, his height above the ground is shrinking at a rate of 2 feet/second.



Throughout this problem, remember to show your work clearly, and include units in your answers.

a. [3 points] How long will the man's shadow (shown in bold in the diagram above) be at the moment when 6 feet of him are above the ground?

Solution: Let s be the length of the shadow. Noticing that the larger and smaller triangles in the picture are similar triangles, we have

$$\frac{30}{28+s} = \frac{6}{s}$$
$$30s = 168 + 6s$$
$$24s = 168$$
$$s = 7.$$

So the length of the shadow is 7 feet at that moment.

Answer: **7 feet**

b. [5 points] At what rate is the length of the man's shadow changing at the moment 6 feet of him are above the ground? Is his shadow growing or shrinking at that moment?

Solution: Let h be the height of the man above the ground, and let s be the length of his shadow. Using similar triangles as above, we have $\frac{30}{28+s} = \frac{h}{s}$ so $30s = 28h + hs$.

Taking derivatives with respect to time t, we find $30\frac{ds}{dt} = 28\frac{dh}{dt} + h\frac{ds}{dt} + s\frac{dh}{dt}$.

So at the moment when $h = 6$, we have

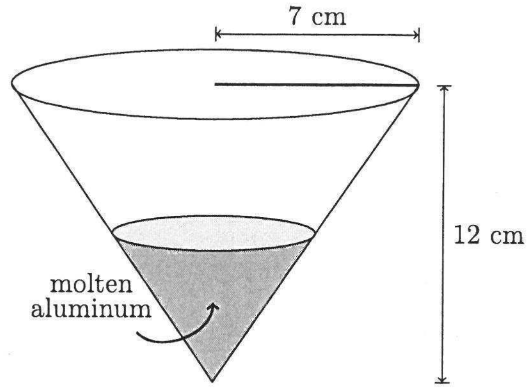
$$30\left.\frac{ds}{dt}\right|_{h=6} = 28(-2) + 6\left.\frac{ds}{dt}\right|_{h=6} + 7(-2)$$
$$24\left.\frac{ds}{dt}\right|_{h=6} = -70$$
$$\left.\frac{ds}{dt}\right|_{h=6} = \frac{-70}{24} = -\frac{35}{12} \approx -2.917$$

So at that moment, the shadow is shrinking at a rate of about 2.917 feet/second.

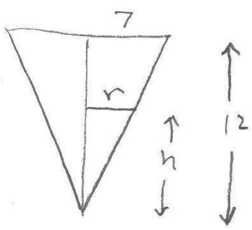
Answer: The man's shadow is (circle one) GROWING [SHRINKING]

at a rate of $\frac{35}{12}$ **(about 2.917) feet/second** .

2. [9 points] Uri is filling a cone with molten aluminum. The cone is upside-down, so the "base" is at the top of the cone and the vertex at the bottom, as shown in the diagram. The base is a circular disk with radius 7 cm and the height of the cone is 12 cm.
Recall that the volume of a cone is $\frac{1}{3}Ah$, where A is the area of the base and h is the height of the cone (i.e., the vertical distance from the vertex to the base). (Note that the diagram may not be to scale.)



a. [3 points] Write a formula in terms of h for the volume V of molten aluminum, in cm^3, in the cone if the molten aluminum in the cone reaches a height of h cm.



Similar triangles:

$$\frac{r}{h} = \frac{7}{12} \Rightarrow r = \frac{7}{12}h$$

So $V = \frac{1}{3}Ah = \frac{1}{3}(\pi r^2)h = \frac{1}{3}\pi\left(\frac{7}{12}h\right)^2 h$

Answer: $V = \frac{49\pi}{432}h^3$

b. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm^3/sec, at which Uri is pouring molten aluminum into the cone at that moment?

$$\frac{dV}{dt} = \frac{49\pi}{432} \cdot 3h^2 \frac{dh}{dt} = \frac{49\pi \cdot 3}{432}(11)^2(3) =$$

Answer: $\frac{5929}{48}\pi \approx 388$ cm^3/sec

c. [3 points] The height of molten aluminum is rising at 3 cm/sec at the moment when the molten aluminum in the cone has reached a height of 11 cm. What is the rate, in cm^2/sec, at which the area of the top surface of the molten aluminum is increasing at that moment?

Area of top $= A = \pi r^2 = \pi\left(\frac{7}{12}h\right)^2 = \frac{49\pi}{144}h^2$

So $\frac{dA}{dt} = \frac{49\pi}{144} \cdot 2h\frac{dh}{dt} = \frac{49\pi}{144}(2)(11)(3) =$

Answer: $\frac{539}{24}\pi \approx 70.55$ cm^2/sec