7. [16 points] Janet is an artist who produces and sells prints of her artwork. If Janet sells her prints for \$17 each, then she will sell 340 prints. Janet is considering whether she should change the price. She takes a survey and concludes that for each price increase of 75 cents, she will sell 10 fewer prints.

a. [4 points] Find a formula for Janet's revenue, $R(x)$, in terms of x, the number of 75 cent price increases.

b. [4 points] Janet plans to produce exactly the number of prints that her survey predicts she will sell. Her costs include \$2 per print, along with \$500 in fixed costs. Find a formula for $C(x)$, Janet's total costs, in terms of x, the number of 75 cent price increases.

c. [8 points] Use the methods of calculus to determine what price Janet should set for her prints if she wants to maximize her profit.

3. [12 points]
Scott is having a graduation party, and his mom wants to order individual cakes for the guests. Each cake is a right circular cylinder with radius R centimeters, height H centimeters, and volume 250 cubic centimeters. In addition,

- there is a fixed cost of \$3 per cake;
- the entire side of the cake will have maize icing with blue candy "M"s, which costs \$0.02 per square centimeter; and
- the entire top of the cake will have blue icing, which costs \$0.01 per square centimeter.

Recall that the volume of a right circular cylinder with radius R and height H is $V = \pi R^2 H$.

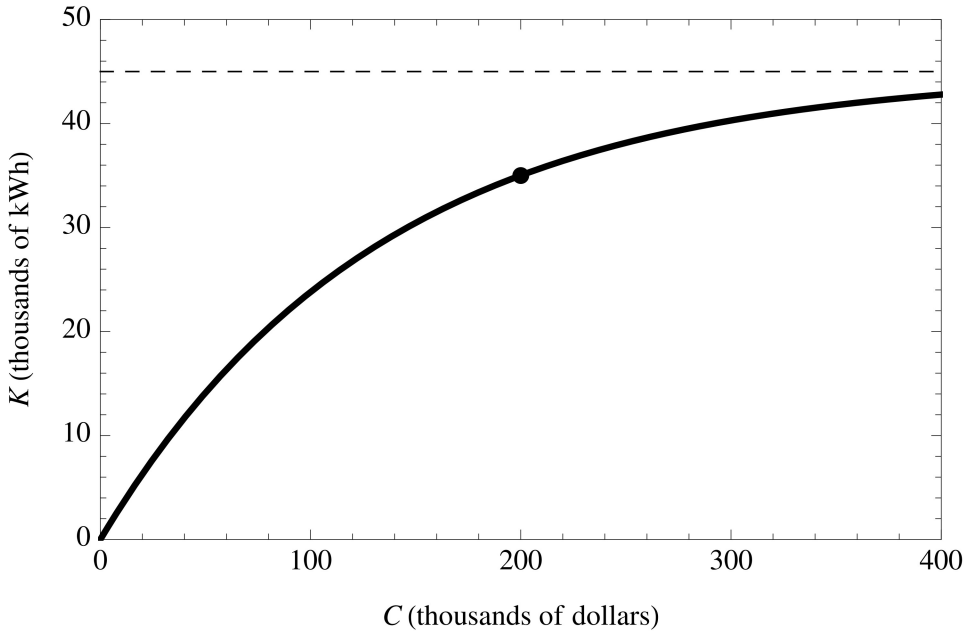
a. [4 points] Find a formula for the cost C of one cake, in terms of its radius R.

b. [8 points] What radius and height should Scott's mom choose for the cakes if she wishes to minimize her costs? What is the minimum price for one cake? (To get credit, you must fully justify your answer using algebraic work.)

radius = _____ *height* = _____ *cost* = _____

8. [10 points] Farmer Fred is designing a fence next to his barn for his grass-fed herd of cattle. The fence will be rectangular in shape with wooden fence on three sides and a chain link fence on the side closest to his barn. The wooden fence costs \$6 per foot and the chain link fence costs \$3 per foot. If he wants the fenced area to be 40,000 square feet, what should the dimensions of his fence be in order to minimize his total cost?

4. [14 points] Business owner Abbey Alexander is constructing a building for her latest business. Abbey wants her building to be energy efficient in order to save money on utility costs. Abbey has been given the following graph to help her decide on how much to spend on improvements.



In the graph, K is the expected savings in thousands of kilowatt hours (kWh) per year if Abbey spends C thousand dollars on energy-efficiency improvements. The dark point on the curve is (200, 35) and the dotted line is a horizontal asymptote at $K = 45$.

a. [5 points] Write a function of the form $K = a(1 - e^{-bC})$ for the curve in the graph above.

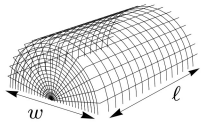
b. [3 points] The current price of energy from Abbey's power company is \$250 per thousand kWh. Assuming this price stays constant, write a function $F(C)$ which gives Abbey's total savings (in thousands of dollars) on utility costs over the first 20 years.

c. [6 points] If Abbey spends C thousand dollars on energy-efficiency improvements, her net monetary savings, N, over 20 years, is given by the formula

$$N = F(C) - C$$

where $F(C)$ is from part **b.** How much should Abbey spend on energy-efficiency improvements in order to maximize her net monetary savings over the first 20 years? Be sure to justify your answer.

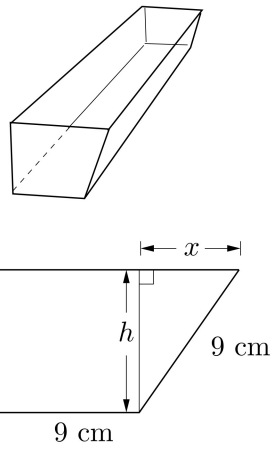
1. [15 points] A hoophouse is an unheated greenhouse used to grow certain types of vegetables during the harsh Michigan winter. A typical hoophouse has a semi-cylindrical roof with a semi-circular wall on each end (see figure to the right). The growing area of the hoophouse is the rectangle of length ℓ and width w (each measured in feet) which is covered by the hoophouse. The cost of the semi-circular walls is \$0.50 per square foot and the cost of the roof, which varies with the side length ℓ, is $\$1 + 0.001\ell$ per square foot.



a. [4 points] Write an equation for the cost of a hoophouse in terms of ℓ and w. *(Hint: The surface area of a cylinder of height ℓ and radius r, not including the circles on each end, is $A = 2\pi r\ell$.)*

b. [11 points] Find the dimensions of the least expensive hoophouse with 8000 square feet of growing area.

2. [13 points] A rain gutter attaches to the edge of a roof and collects the rain that falls on the roof. A common gutter design is shown in the figure to the right, and has a trapezoidal cross-section (also shown). In this problem we consider a gutter with base and side length 9 cm, as shown.

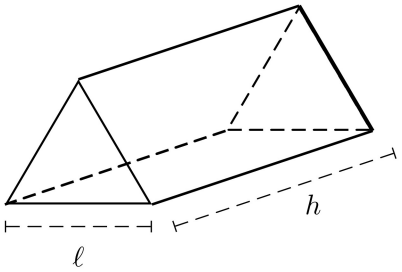


a. [1 point] Write an equation which relates the length x to the height h.

b. [4 points] Using your equation from (a), write an equation for the cross-sectional area of the gutter as a function of the length x (note that the area is the sum of a rectangular and right-triangular region).

c. [8 points] Find the length x that gives the maximum cross-sectional area. Be sure to show work that demonstrates that you have found the maximum.

3. [12 points] Consider the prism with equilateral triangles of side length ℓ centimeters for ends and a length of h centimeters, illustrated below. The volume of this prism is $\sqrt{3}\,\ell^2 h/4$. You may find it useful to note that the area of an equilateral triangle of side length ℓ is $\sqrt{3}\,\ell^2/4$.

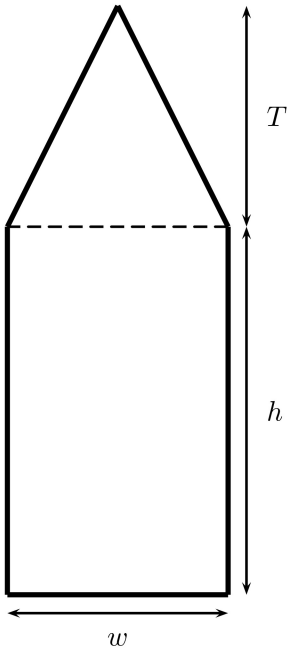


a. [4 points] Give the equation of the surface area of this prism, listing units.

Surface area=_____

b. [8 points] If the prism has a fixed volume of 16 cm^3, find the values of ℓ and h which minimize the surface area. Clearly justify that you have found the minimum.

6. [10 points] Consider a window the shape of which is a rectangle of height h surmounted by a triangle having a height T that is two times the width w of the rectangle (see the figure below which is not drawn to scale). If the total area of the window is 5 square feet, determine the dimensions of the window which minimize the perimeter.

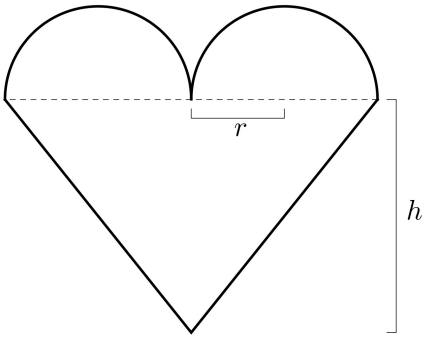


7. [12 points] For Valentine's Day, Jason decides to make a heart-shaped cookie for Sophie to try to win her over. Being mathematically-minded, the only kind of heart that Jason knows how to construct is composed of two half-circles of radius r and an isosceles triangle of height h, as shown below.

Jason happens to know that Sophie's love for him will be determined by the dimensions of the cookie she receives; if given a cookie as described above, her love L will be

$$L = hr^2,$$

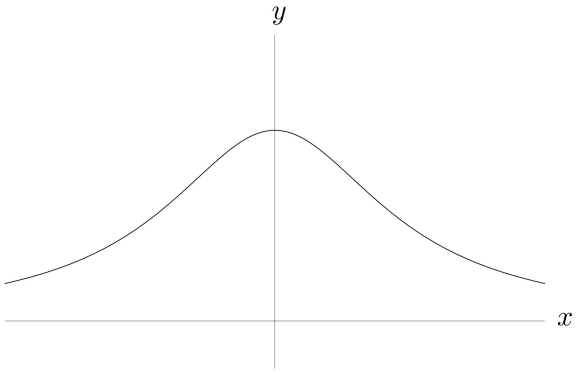
where r and h are measured in centimeters and L is measured in pitter-patters, a standard unit of affection. If Jason wants to make a cookie whose area is exactly 300cm^2, what should the dimensions be to maximize Sophie's love?



5. [12 points] A rectangle has one side on the x-axis and two vertices on the curve

$$y = \frac{36}{9 + x^2}.$$

This curve is graphed below. Find the x- and y- coordinates of all four vertices of the rectangle with largest area. You must show that your vertices maximize the area of the rectangle.

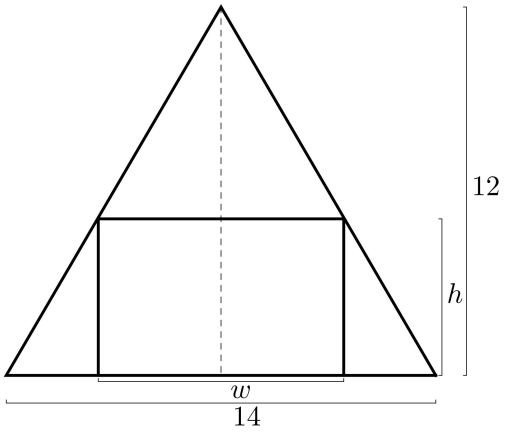


8. [13 points] Two smokestacks d miles apart deposit soot on the ground between them. The concentration of the combined soot deposits on the line joining them, at a distance x from one stack, is given by

$$S = \frac{c}{x^2} + \frac{k}{(d-x)^2}$$

where c and k are positive constants which depend on the quantity of smoke each stack is emitting. If $k = 27c$, find the x-value of the point on the line joining the stacks where the concentration of the deposit is a minimum. Justify that the point you found is actually a global minimum.

2. [5 points] Caleb has an attic apartment, and his bedroom has a triangular wall that is 14 feet wide and 12 feet tall at its tallest point. He wants to build a rectangular bookcase to put against the wall, as shown to the right. He is trying to maximize the area of the front of the bookcase.

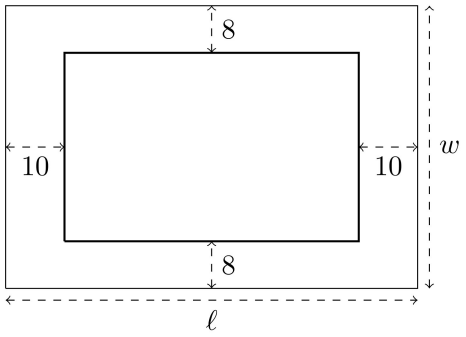


a. [3 points] If the bookcase has width w and height h, write a formula relating w and h.

b. [2 points] Using your answer from (a), find an expression for the area of the front of the bookcase in terms of the variable h.

3. [4 points] Suppose $g(x) = x^{2x}$. Write an explicit expression for $g'(5)$ using the limit definition of the derivative. Your expression should not contain the letter "g". Do not evaluate your expression.

1. [7 points] Liam wants to build a rectangular swimming pool behind his new house. The pool will have an area of 1600 square feet. He will have 8-foot wide decks on two sides of the pool and 10-foot wide decks on the other two sides of the pool (see the diagram below).



a. [4 points] Let ℓ and w be the length and width (in feet) of the pool area including the decks as shown in the diagram. Write a formula for ℓ in terms of w.

$\ell =$ _____

b. [3 points] Write a formula for the function $A(w)$ which gives the total area (in square feet) of the pool **and** the decks in terms of only the width w. Your formula should not include the variable ℓ. (This is the function Liam would minimize in order to find the minimum area that his pool and deck will take up in his yard. You do not need to do the optimization in this case.)

$A(w) =$ _____

8. [12 points] For Thanksgiving, Bert is trying to make a festive feast table using fall-colored cloth and other accessories. The cloth costs \$0.25 per square foot and the accessories are \$0.50 each. He decides the impact of the festive table, I, is a function of the number of square feet of cloth, c, that he uses and the number of accessories, a, that he uses. This relationship is given by

$$I = c\left(\frac{1}{2}a - 3\right)^2.$$

Bert has a total budget of \$9 for the cloth and accessories.

a. [2 points] Write an equation which expresses that the total cost of the cloth plus the accessories for the festive table is \$9.

b. [10 points] Use your answer from **(a)** to find the maximum impact of the festive table that is possible for \$9, as well as how many accessories and how much cloth is needed to achieve the maximum impact. Be sure to show your answer is indeed the maximum.

maximum impact: $I =$ _____

$c =$ _____

$a =$ _____

4. [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that t seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where k and c are nonzero constants.

a. [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of t. The constants k and c may appear in your answer.

Answer: $v(t) =$ _____

b. [2 points] Find a formula for $v'(t)$. The constants k and c may appear in your answer.

Answer: $v'(t) =$ _____

c. [3 points] What is the maximum vertical acceleration experienced by the captain? The constants k and c may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

Answer: Max vertical acceleration: _____

9. [10 points] After a long, cold winter, a ship's captain sails across Lake Michigan to Chicago. Upon arrival, the captain hosts a party on board to celebrate the arrival of spring. The party begins at exactly 6 pm and ends at exactly midnight. Let $N(t)$ be the noise level, in decibels, of the ship captain's party t hours after it begins. During the party, a formula for $N(t)$ is given by

$$N(t) = 0.5t^4 - 4t^3 + 7t^2 + 60.$$

a. [8 points] Find the exact values of t that minimize and maximize $N(t)$ on the interval $[0, 6]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema.

(For each answer blank below, write NONE in the answer blank if appropriate.)

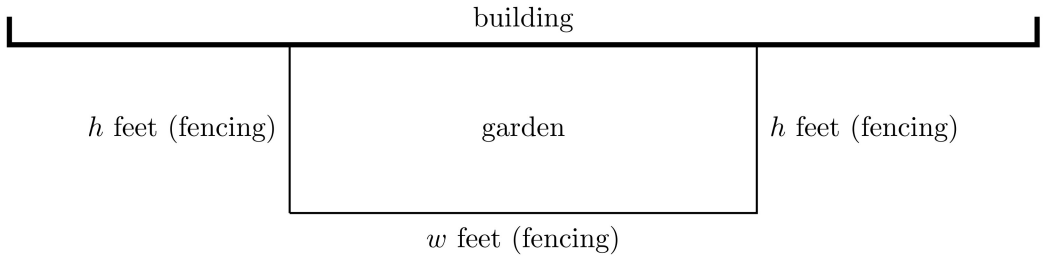
Answer: Global min(s) at exactly $t =$ _____

Answer: Global max(es) at exactly $t =$ _____

b. [2 points] How loud does the captain's party get? *Remember to include units.*

Answer: _____

4. [12 points] Researchers are constructing a rectangular garden adjacent to their building. The garden will be bounded by the building on one side and by a fence on the other three sides. (See diagram below.) The fencing will cost them \$5 per linear foot. In addition, they will also need topsoil to cover the entire area of the garden. The topsoil will cost \$4 per square foot of the garden's area.
Assume the building is wider than any garden the researchers could afford to build.



a. [5 points] Suppose the garden is w feet wide and extends h feet from the building, as shown in the diagram above. Assume it costs the researchers a total of \$250 for the fencing and topsoil to construct this garden. Find a formula for w in terms of h.

Answer: $w =$ _____

b. [3 points] Let $A(h)$ be the total area (in square feet) of the garden if it costs \$250 and extends h feet from the building, as shown above. Find a formula for the function $A(h)$. The variable w should not appear in your answer.

(Note that $A(h)$ is the function one would use to find the value of h maximizing the area. You should not do the optimization in this case.)

Answer: $A(h) =$ _____

c. [4 points] In the context of this problem, what is the domain of $A(h)$?

Answer: _____