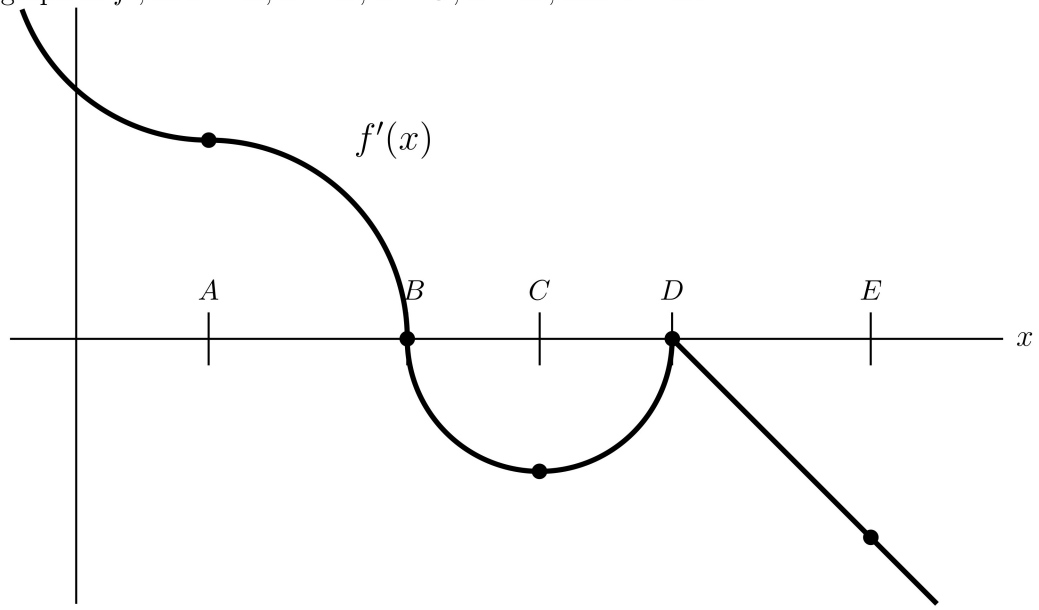
6. [12 points] The *derivative* of a function f is graphed below. Five points are marked on the graph of f', at $x = A$, $x = B$, $x = C$, $x = D$, and $x = E$.



For each of the following, circle ALL answers which are correct. Each part has at least one correct answer. Pay careful attention to whether each question is asking about f, f', or f''.

a. [2 points] The function f' has a local minimum when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

b. [2 points] The function f is increasing when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

c. [2 points] The function f has a critical point when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

d. [2 points] The global maximum of f on the interval $A \leq x \leq E$ occurs when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

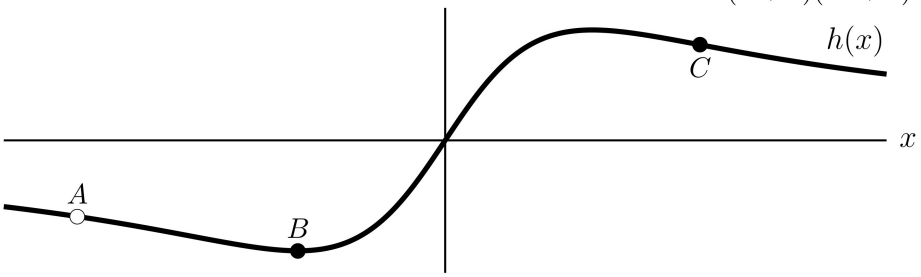
e. [2 points] The function f has an inflection point when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

f. [2 points] The function f'' is undefined when _____.

$x = A$ $x = B$ $x = C$ $x = D$ $x = E$

8. [13 points] Below, there is a graph of the function $h(x) = \dfrac{2x^2 + 10x}{(x+5)(x^2+4)}$.



a. [3 points] The point A is a hole in the graph of h. Find the x- and y-coordinates of A.

b. [5 points] The point B is a local minimum of h. Find the x- and y-coordinates of B.

c. [5 points] The point C is an inflection point of h. Find the x- and y-coordinates of C.

2. [14 points] The table for the *derivative* of a function h with continuous first derivative is given below. Assume that between each consecutive value of x, the derivative h' is either increasing or decreasing. For each statement below, indicate whether the statement is true, false, or cannot be determined from the information given. No partial credit will be given.

x	-4	-3	-2	-1	0	1	2	3	4
$h'(x)$	2	3	1	-3	-4	-2	0	2	1

a.) The function h has a local maximum on the interval $-2 < x < -1$.

True False Not enough information

b.) The function h is negative on the interval $-1 < x < 1$.

True False Not enough information

c.) The function h is concave up on the interval $0 < x < 4$.

True False Not enough information

d.) The function h is decreasing on the interval $-3 < x < -2$.

True False Not enough information

e.) The function h has an inflection point on the interval $-1 < x < 1$.

True False Not enough information

f.) The derivative function, h', has a critical point at $x = 2$.

True False Not enough information

g.) The second derivative function, h'', is positive on the interval $0 < x < 3$.

True False Not enough information

7. [15 points] Suppose a is a positive constant and

$$f(x) = 2x^3 - 3ax^2.$$

a. [10 points] Find the absolute maximum and minimum values of $f(x)$ on the closed interval $[-a, \frac{3}{2}a]$. Specify all x values where the maximum and minimum value are achieved.

b. [5 points] Find all inflection points of $f(x)$.

2. [13 points] The U-value of a wall of a building is a positive number related to the rate of energy transfer through the wall. Walls with a lower U-value keep more heat in during the winter than ones with a higher U-value. Consider a wall which consists of two materials, material A with U-value a and material B with U-value b. The U-value of the wall w is given by

$$w = \frac{ab}{b+a}.$$

Considering a as a constant, we can think of w as a function of b, $w = u(b)$.

a. [4 points] Write the limit definition of the derivative of $u(b)$.

b. [4 points] Calculate $u'(b)$. (You do **not** need to use the limit definition of the derivative for your calculation.)

c. [5 points] Find the x- and y-coordinates of the global minimum and maximum of $u(b)$ for b in the interval $[1, 2]$. Your answer may involve the parameter a. [Recall that $a, b > 0$.]

Global minimum on $[1, 2]$: _____

Global maximum on $[1, 2]$: _____

3. [12 points] Representative values of the derivative of a function $f(x)$ are shown in the table below. Assume $f'(x)$ is a continuous function and that the values in the table are representative of the behavior of $f'(x)$.

x	0	0.5	1	1.5	2	2.5	3
$f'(x)$	1	0.3	0	-0.1	-0.15	-0.12	-0.10

a. [6 points] Estimate the location of the global maximum and minimum of $f(x)$ on the closed interval $[0, 3]$. Justify your answers based on the data in the table.

b. [6 points] Can you tell from these data if $f(x)$ has any inflection points? If so, estimate the location of any inflection points and indicate how you know their locations. If not, explain why not.

9. [12 points] Suppose $w(x)$ is an everywhere differentiable function which satisfies the following conditions:

- $w'(0) = 0$.
- $w'(x) > 0$ for $x > 0$.
- $w'(x) < 0$ for $x < 0$.

Let $f(t) = t^2 + bt + c$ where b and c are positive constants with $b^2 > 4c$. Define $L(t) = w(f(t))$.

a. [2 points] Compute $L'(t)$. Your answer may involve w and/or w' and constants b and c.

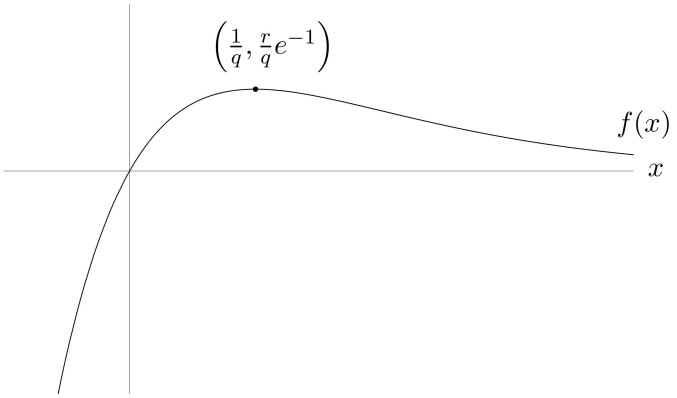
b. [4 points] Using your answer from (a), find the critical points of $L(t)$ in terms of the constants b and c.

c. [6 points] Classify each critical point you found in (b). Be sure to fully justify your answer.

8. [16 points] Below is the graph of the function

$$f(x) = rxe^{-qx},$$

where r and q are constants. Assume that both r and q are greater than 1. The function $f(x)$ passes through the origin and has a local maximum at the point $P = \left(\frac{1}{q}, \frac{r}{q}e^{-1}\right)$, as shown in the graph.

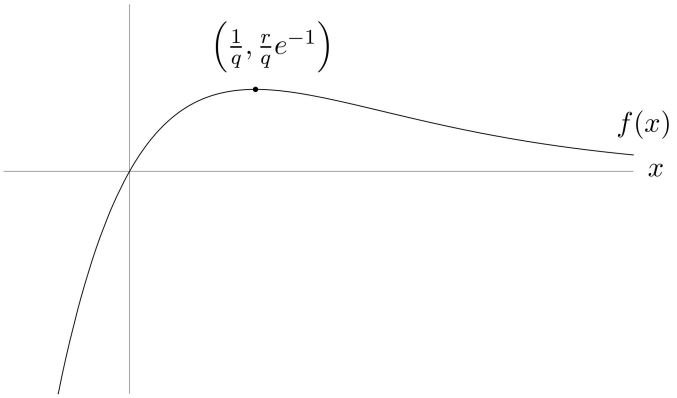


a. [4 points] Justify, using either the first-derivative test or second-derivative test, that the point P is a local maximum.

b. [2 points] What are the x-coordinates of the global maximum and minimum of $f(x)$ on the domain $[0, 1]$? (If $f(x)$ does not have a global maximum on this domain, say "no global maximum", and similarly if $f(x)$ does not have a global minimum.)

c. [2 points] What are the x-coordinates of the global maximum and minimum of $f(x)$ on the domain $(-\infty, \infty)$? (If $f(x)$ does not have a global maximum on this domain, say "no global maximum", and similarly if $f(x)$ does not have a global minimum.)

8. (continued) For your convenience, the graph of $f(x)$ is repeated below.



d. [4 points] Suppose that $g(x)$ is a function with $g'(x) = f(x)$. Find x-values of all local maxima and minima of $g(x)$. Justify that each maximum you find is a maximum and each minimum is a minimum.

e. [4 points] If $g(x)$ is as in part (d), for which x-values does $g(x)$ have inflection points? Show that these x-values are indeed inflection points.

5. [14 points] The function f is has a continuous second derivative on the interval $10 \leq x \leq 19$. Some values of its derivative function f' are given in the table below.

x	10	11	12	13	14	15	16	17	18	19
$f'(x)$	-34	-3	-1	-2	-3	31	62	70	66	37

a. [4 points] f has exactly one inflection point on the interval $15 \leq x \leq 19$. Given the information provided, give the smallest x interval on which this inflection point is guaranteed to lie, making it clear whether your endpoints are included.

b. [8 points] f has exactly four critical points, with x-values 11.2, 11.7, 12.6, and 14.2, respectively. Classify each point as a local minimum, a local maximum, or neither, given that f has either a local maximum or a local minimum at $x = 11.2$. For each point below, circle only one option.

At $x = 11.2$, f has **a local maximum** **a local minimum**

At $x = 11.7$, f has **a local maximum** **a local minimum** **neither**

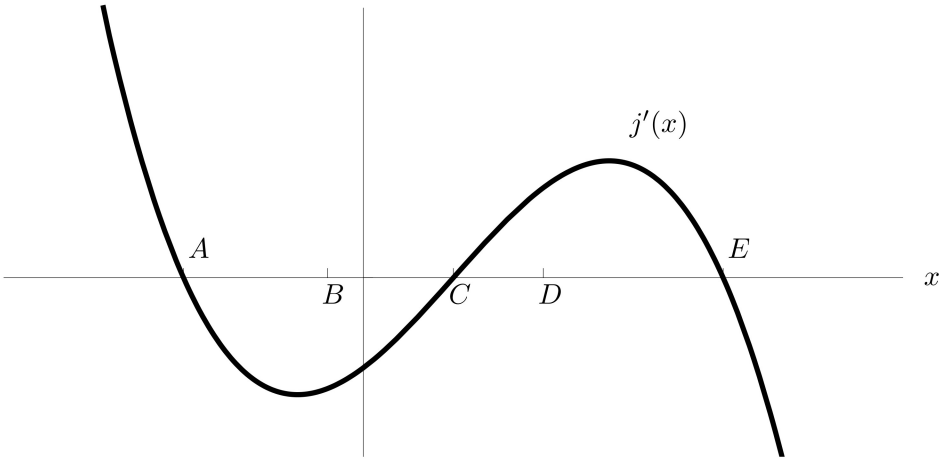
At $x = 12.6$, f has **a local maximum** **a local minimum** **neither**

At $x = 14.2$, f has **a local maximum** **a local minimum** **neither**

c. [2 points] Is there at least one inflection point on the interval $11 < x < 12$? (Circle one.)

Yes **No** **Not possible to determine**

1. [12 points] Consider the graph of $j'(x)$ given here. Note that this is not the graph of $j(x)$.



For each of (a)-(f) below, list **all** x-values labeled on the graph which satisfy the given statement in the blank provided. If the statement is not true at any of the labeled x-values, write "NP". You do not need to show your work. No partial credit will be given on each part of this problem.

(a) The function $j(x)$ has a local minimum at $x =$_____.

(b) The function $j(x)$ has a local maximum at $x =$_____.

(c) The function $j(x)$ is concave up at $x =$_____.

(d) The function $j(x)$ is concave down at $x =$_____.

(e) The function $j'(x)$ has a critical point at $x =$_____.

(f) The function $j''(x)$ is greatest at $x =$_____.

9. [10 points] The function $f(x)$ is twice-differentiable. Some values of f and f' are given in the following table. In addition, it is known that $f''(x)$ is positive.

x	0	1	2	3	4
$f(x)$	7	6	7	9	12
$f'(x)$	-2	$\frac{1}{2}$	1	2	4

No partial credit will be given on any part of this problem.

a. [4 points] **Circle** any statement which is true, and **draw a line through** any statement which is false.

(i.) For some value of x with $0 < x < 1$, f has a critical point.

(ii.) For some value of x with $1 < x < 2$, f has a critical point.

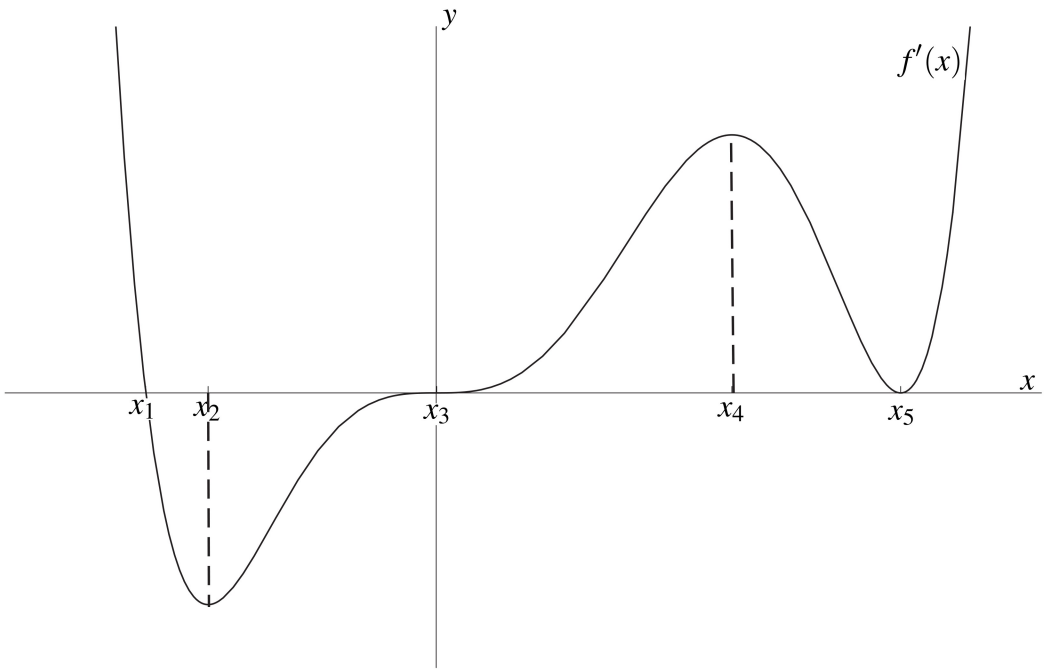
(iii.) For some value of x with $2 < x < 3$, f has a critical point.

(iv.) For some value of x with $3 < x < 4$, f has a critical point.

b. [3 points] If possible, find the global minimum value of $f(x)$ on the closed interval $[0, 4]$. (Give the y-coordinate, not the x-coordinate.) Do not give an approximation. If it is not possible to find it exactly, write "IT IS NOT POSSIBLE TO FIND IT EXACTLY."

c. [3 points] If possible, find the global maximum value of $f(x)$ on the closed interval $[0, 4]$. (Give the y-coordinate, not the x-coordinate.) Do not give an approximation. If it is not possible to find it exactly, write "IT IS NOT POSSIBLE TO FIND IT EXACTLY."

1. (12 points) The graph below is a plot of $f'(x)$ (the *derivative* of f). Use the graph to answer questions about the function f.



(a) What are the critical points of f? _____

(b) For what value(s) of x does f have a local maximum? _____

(c) For what value(s) of x does f have a local minimum? _____

(d) What are the inflection points of f? _____

(e) On what interval(s) is f concave up? _____

(f) If f is a polynomial, what is the minimal degree of f? _____

2. (12 points) Suppose a is a positive (non-zero) constant, and consider the function

$$f(x) = \frac{1}{3}x^3 - 4a^2x.$$

Determine all maxima and minima of f in the interval $[-3a, 5a]$. For each, specify whether it is global or local.

4. [10 points] Let $h(x)$ be a twice differentiable function defined for all real numbers x. (So h is differentiable and its derivative h' is also differentiable.)
Some values of $h'(x)$, the derivative of h are given in the table below.

x	-8	-6	-4	-2	0	2	4	6	8
$h'(x)$	3	7	0	-3	-5	-4	0	-2	6

For each of the following, circle all the correct answers.
Circle "NONE OF THESE" if none of the provided choices are correct.

a. [2 points] Circle all the intervals below in which $h(x)$ must have a critical point.

$-8 < x < -6$ $-6 < x < -2$ $-2 < x < 2$ $2 < x < 6$ $6 < x < 8$

NONE OF THESE

b. [2 points] Circle all the intervals below in which $h(x)$ must have a local extremum (i.e. a local maximum or a local minimum).

$-8 < x < -6$ $-6 < x < -2$ $-2 < x < 2$ $2 < x < 6$ $6 < x < 8$

NONE OF THESE

c. [2 points] Circle all the intervals below in which $h(x)$ must have an inflection point.

$-8 < x < -4$ $-4 < x < 0$ $0 < x < 4$ $2 < x < 6$ $4 < x < 8$

NONE OF THESE

d. [2 points] Circle all the intervals below which must contain a number c such that $h''(c) = 2$.

$-8 < x < -6$ $-4 < x < -2$ $-2 < x < 0$ $2 < x < 4$ $6 < x < 8$

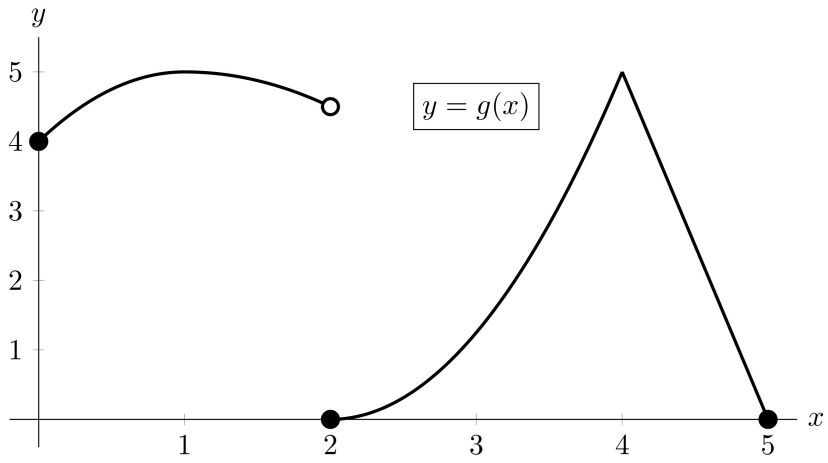
NONE OF THESE

e. [2 points] Suppose that $h''(x) < 0$ for $x < -8$, and $h(-8) = 7$. Circle all the numbers below which could equal the value of $h(-10)$.

-2 -1 0 1 2

NONE OF THESE

6. [14 points] The entire graph of a function $g(x)$ is shown below. Note that the graph of $g(x)$ has a horizontal tangent line at $x = 1$ and a sharp corner at $x = 4$.



For each of the questions below, circle <u>all</u> of the available correct answers.
(Circle NONE OF THESE if none of the available choices are correct.)

a. [2 points] At which of the following values of x does $g(x)$ appear to have a critical point?

$x = 1$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ $x = 4$ $\quad$ NONE OF THESE

b. [2 points] At which of the following values of x does $g(x)$ attain a local maximum?

$x = 1$ $\quad$ $x = 2$ $\quad$ $x = 3$ $\quad$ $x = 4$ $\quad$ NONE OF THESE

c. [6 points] Let $L(x)$ be the local linearization of $g(x)$ near $x = 3$. Circle all of the statements that are true.

$L(3) > g(3)$	$L(2.5) > g(2.5)$	$L(0) > g(0)$
$L(3) = g(3)$	$L(2.5) = g(2.5)$	$L(0) = g(0)$
$L(3) < g(3)$	$L(2.5) < g(2.5)$	$L(0) < g(0)$
$L'(3) > g'(3)$	$L'(2.5) > g'(2.5)$	$L(5) > g(5)$
$L'(3) = g'(3)$	$L'(2.5) = g'(2.5)$	$L(5) = g(5)$
$L'(3) < g'(3)$	$L'(2.5) < g'(2.5)$	$L(5) < g(5)$

NONE OF THESE

d. [2 points] On which of the following intervals does $g(x)$ satisfy the hypotheses of the Mean Value Theorem?

$[0, 2]$ $\quad$ $[0, 4]$ $\quad$ $[3, 5]$ $\quad$ $[4, 5]$ $\quad$ NONE OF THESE

e. [2 points] On which of the following intervals does $g(x)$ satisfy the conclusion of the Mean Value Theorem?

$[0, 2]$ $\quad$ $[0, 4]$ $\quad$ $[3, 5]$ $\quad$ $[4, 5]$ $\quad$ NONE OF THESE