Math 115 — Practice for Exam 2

Generated February 22, 2017

Name: **SOLUTIONS**

Instructor: ______________________________ Section Number: __________

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1. This exam has 11 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Fall 2016	2	11		6	
Fall 2010	2	6	cactus cube	10	
Winter 2011	2	6		15	
Winter 2016	2	8		14	
Winter 2012	2	5		12	
Fall 2012	2	4		13	
Winter 2013	2	4		11	
Winter 2014	2	10		10	
Fall 2014	2	1		5	
Winter 2015	2	3		8	
Winter 2016	2	10		9	
Total				113	

Recommended time (based on points): 102 minutes

10. [4 points] Let a and b be constants. Consider the curve $\mathcal{C}$ defined by the equation

$$\cos(ax) + by\ln(x) = 3 + y^3.$$

Find a formula for $\frac{dy}{dx}$ in terms of x and y. The constants a and b may appear in your answer. To earn credit for this problem, you must compute this by hand and show every step of your work clearly.

Solution: We use implicit differentiation.

$$\begin{aligned}\frac{d}{dx}(\cos(ax) + by\ln(x)) &= \frac{d}{dx}(3 + y^3)\\ -a\sin(ax) + \frac{by}{x} + b\ln(x)\frac{dy}{dx} &= 3y^2\frac{dy}{dx}\\ \frac{dy}{dx} &= \frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)}\end{aligned}$$

Answer: $\frac{dy}{dx} = \frac{\frac{by}{x} - a\sin(ax)}{3y^2 - b\ln(x)}$

11. [6 points] Let $h(x) = x^x$. For this problem, it may be helpful to know the following formulas:

$$h'(x) = x^x(\ln(x) + 1) \qquad \text{and} \qquad h''(x) = x^x\left(\frac{1}{x} + (\ln(x) + 1)^2\right).$$

a. [2 points] Write a formula for $p(x)$, the local linearization of $h(x)$ near $x = 1$.

Solution: $h(1) = 1$ and $h'(1) = 1^1(\ln(1) + 1) = 1$, so $p(x) = 1 + 1\cdot(x - 1) = x$.

Answer: $p(x) =$ x

b. [4 points] Write a formula for $u(x)$, the quadratic approximation of $h(x)$ at $x = 1$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: $h''(1) = 1(1 + (0 + 1)^2) = 2$, so $u(x) = 1 + (x - 1) + \frac{2}{2}(x - 1)^2 = x^2 - x - 1$.

Answer: $u(x) =$ $1 + (x - 1) + (x - 1)^2$ $(= x^2 - x - 1)$

6. [10 points] Calvin is stuck in the desert, and he needs to build a cube out of cactus skins to hold various supplies. He wants his cube to have a volume of 8.1 cubic feet, but he needs to figure out the side length to cut the cactus skins the right size. He has forgotten his trusty calculator, so he decides to figure out the side length of his cube using calculus.

a. [5 points] Find a local linearization of the function $f(x) = (x+8)^{1/3}$ at $x = 0$.

Solution: The derivative is $f'(x) = \frac{1}{3}(x+8)^{-2/3}$. To find the local linearization we compute $f'(0) = \frac{1}{3}(8)^{-2/3} = \frac{1}{12}$ and $f(0) = 2$. The equation for the tangent line to f at $x = 0$ is $y - 2 = \frac{1}{12}x$. So the local linearization of f near $x = 0$ is

$$L(x) = \frac{1}{12}x + 2.$$

b. [3 points] Use your linearization to approximate $(8.1)^{1/3}$.

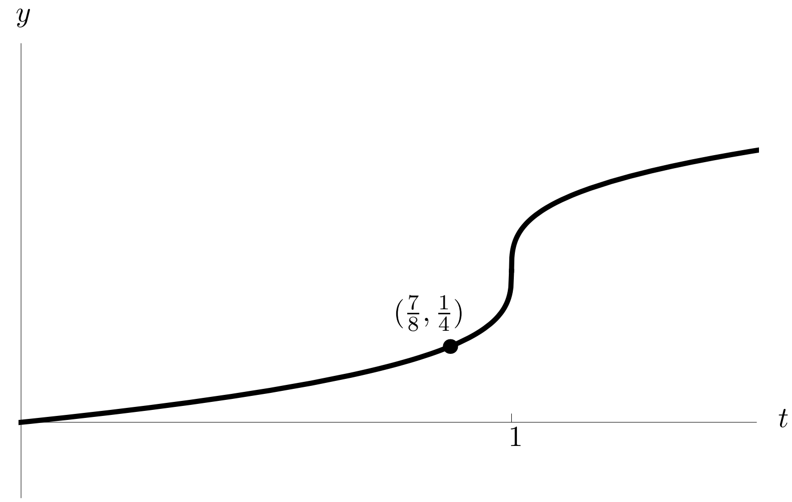
Solution: We need to approximate $(8.1)^{1/3} = f(0.1)$. According to our local linearization,

$$f(0.1) \approx L(0.1) = \frac{1}{12}(0.1) + 2 = \frac{241}{120}.$$

c. [2 points] Should your approximation from part **b.** be an over-estimate or an under-estimate? Why?

Solution:
The second derivative of f is $f''(x) = -\frac{2}{9}(x+8)^{-5/3}$. For values of x near 0, the second derivative will be negative which means f is concave down near 0. This means our estimate is an overestimate.

6. [15 points] Given below is the graph of a function $h(t)$. Suppose $j(t)$ is the local linearization of $h(t)$ at $t = \frac{7}{8}$.



a. [5 points] Given that $h'(\frac{7}{8}) = \frac{2}{3}$, find an expression for $j(t)$.

Solution: The local linearization is the tangent line to the curve. We know this line has slope $h'(\frac{7}{8}) = \frac{2}{3}$ and it goes through the point $(\frac{7}{8}, \frac{1}{4})$, so it has equation

$$y - \frac{1}{4} = \frac{2}{3}(t - \frac{7}{8})$$

using point slope form. Solving for y we have $y = \frac{2}{3}t - \frac{1}{3}$. So $j(t) = \frac{2}{3}t - \frac{1}{3}$. stuff

b. [4 points] Use your answer from (a) to approximate $h(1)$.

Solution: Since $j(t)$ approximates $h(t)$ for t-values near $\frac{7}{8}$, we have

$$h(1) \approx j(1) = \frac{2}{3}(1) - \frac{1}{3} = \frac{1}{3}.$$

c. [3 points] Is the approximation from (b) an over- or under-estimate? Explain.

Solution: The approximation in (b) is an underestimate. The function $h(t)$ is concave up at $t = 7/8$ which means the graph lies above the local linearization for t-values near $7/8$. Since we are using the local linearization to estimate the function value, our estimate will be less than the actual function value.

d. [3 points] Using $j(t)$ to estimate values of $h(t)$, will the estimate be more accurate at $t = 1$ or at $t = \frac{3}{4}$? Explain.

Solution: The estimate at $t = 3/4$ will be more accurate. This can be seen by drawing the tangent line and measuring the vertical distance between the estimated value and the function value at the t values $3/4$ and 1. The line is much closer to the function at $t = 3/4$ than it is at $t = 1$.

8. [14 points]
Suppose H is a differentiable function such that $H'(w)$ is also differentiable for $0 < w < 10$. Several values of $H(w)$ and of its first and second derivatives are given in the table on the right.

w	1	2	3	5	8
$H(w)$	6.3	5.4	5.2	4.8	0.7
$H'(w)$	-1.5	-0.4	-0.1	-0.6	-2.1
$H''(w)$	1.6	0.9	0	-0.8	-0.4

Assume that between each pair of consecutive values of w shown in the table, each of $H'(w)$ and $H''(w)$ is either always strictly decreasing or always strictly increasing.
Remember to show your work carefully.

a. [3 points] Use an appropriate linear approximation to estimate $H(5.2)$.

Solution: For w near 5, local linearization gives $H(w) \approx H(5) + H'(5)(w - 5)$, so

$$H(5.2) \approx H(5) + H'(5)(5.2 - 5) = 4.8 - 0.6(0.2) = 4.8 - 0.12 = 4.68.$$

Answer: $H(5.2) \approx$ 4.68

b. [5 points] Let $J(w)$ be the local linearization of H near $w = 2$, and let $K(w)$ be the local linearization of H near $w = 3$. Which of the following statements must be true? Circle all of the statements that must be true, or circle "NONE OF THESE".

$J(2) > H(2)$	$J(2.5) > H(2.5)$	**[$K(3.5) > H(3.5)$]**
[$J(2) = H(2)$]	$J(2.5) = H(2.5)$	$K(3.5) = H(3.5)$
$J(2) < H(2)$	**[$J(2.5) < H(2.5)$]**	$K(3.5) < H(3.5)$
$J'(2) > H'(2)$	$K(2.5) > H(2.5)$	**[$K'(3.5) > H'(3.5)$]**
[$J'(2) = H'(2)$]	$K(2.5) = H(2.5)$	$K'(3.5) = H'(3.5)$
$J'(2) < H'(2)$	**[$K(2.5) < H(2.5)$]**	$K'(3.5) < H'(3.5)$

NONE OF THESE

c. [3 points] Use the quadratic approximation of $H(w)$ at $w = 1$ to estimate $H(0.9)$. (Recall that a formula for the quadratic approximation $Q(x)$ of a function $f(x)$ at $x = a$ is $Q(x) = f(a) + f'(a)(x - a) + \frac{f''(a)}{2}(x - a)^2$.)

Solution: Let $Q(w)$ be the quadratic approximation of $H(w)$ at $w = 1$.
Then $Q(w) = H(1) + H'(1)(w - 1) + \frac{H''(1)}{2}(w - 1)^2 = 6.3 - 1.5(w - 1) + \frac{1.6}{2}(w - 1)^2$.
So, $H(0.9) \approx Q(0.9) = 6.3 - 1.5(0.9 - 1) + \frac{1.6}{2}(0.9 - 1)^2 = 6.3 + 0.15 + 0.008 = 6.458$.

Answer: $H(0.9) \approx$ 6.458

d. [3 points] Consider the function N defined by $N(w) = H(2w^2 - 10)$, and let $L(w)$ be the local linearization of $N(w)$ at $w = 3$. Find a formula for $L(w)$. Your answer should not include the function names N or H.

Solution: We know that $L(w) = N(3) + N'(3)(w - 3)$.
Note that $N(3) = H(2(3^2) - 10) = H(8) = 0.7$.
To find $N'(3)$, we apply the Chain Rule. In particular, $N'(w) = (4w) \cdot H'(2w^2 - 10)$, so $N'(3) = (4 \cdot 3) \cdot H'(2(3^2) - 10) = 12H'(8) = 12(-2.1) = -25.2$.
Therefore, $L(w) = N(3) + N'(3)(w - 3) = 0.7 - 25.2(w - 3)$.

Answer: $L(w) =$ $0.7 - 25.2(w - 3)$

5. [12 points]

a. [3 points] Find the local linearization $L(x)$ of the function

$$f(x) = (1+x)^k$$

near $x = 0$, where k is a positive constant.

Solution: The derivative is $f'(x) = k(1+x)^{k-1}$, so the slope of the tangent line at $x = 0$ is

$$f'(0) = k.$$

Since $f(0) = 1^k = 1$, the tangent line passes through the point $(0, 1)$. Therefore, the point-slope formula shows that the equation of the tangent line is

$$y = kx + 1.$$

b. [3 points] For which values of k does this local linearization give underestimates of the actual value of $f(x)$? (Show your work.)

Solution: The local linearization gives underestimates of the actual value when $f''(0) > 0$. The second derivative is $f''(x) = k(k-1)(1+x)^{k-2}$, so

$$f''(0) = k(k-1).$$

Since $k > 0$, this is positive when the second factor is positive, which is when $k > 1$.

c. [2 points] Suppose you want to use $L(x)$ to find an approximation of the number $\sqrt{1.1}$. What number should k be, and what number should x be?

Solution: If $k = \frac{1}{2}$ and $x = 0.1$, then $f(0.1) = \sqrt{1.1}$, so $L(1.1)$ gives an approximation of $\sqrt{1.1}$.

d. [2 points] Approximate $\sqrt{1.1}$ using $L(x)$.

Solution: If k and x are as above, then $\sqrt{1.1} \approx L(0.1) = 1.05$.

e. [2 points] What is the error in the approximation from part (d)?

Solution: The error is the actual value minus the approximate value which is $\sqrt{1.1} - 1.05 \approx -0.00119$.

4. [13 points] Let $f(x) = e^{\sin\sqrt{x}}$. Let P be the point on the graph of f at which $x = 4\pi^2 (\approx 39.4784)$.

a. [3 points] Calculate $f'(x)$.

Solution:

$$f'(x) = \left(e^{\sin\sqrt{x}}\right)\left(\cos\sqrt{x}\right)\left(\frac{1}{2}x^{-1/2}\right) = \frac{e^{\sin\sqrt{x}}\cos\sqrt{x}}{2\sqrt{x}}$$

b. [4 points] Find an **exact** formula for the tangent line $L(x)$ to $f(x)$ at P. **Exact** means your answer should not involve any decimal approximations.

Solution:

$$\text{slope} = f'(4\pi^2) = \frac{e^{\sin(2\pi)}\cos(2\pi)}{2\cdot 2\pi} = \frac{1}{4\pi},$$

so $L(x) = \frac{x}{4\pi} + b$, where b is the vertical intercept. When $f(4\pi^2) = e^{sin(2\pi)} = 1$, so $1 = \frac{4\pi^2}{4\pi} + b$, which gives us $b = 1 - \pi$, so

$$L(x) = \frac{x}{4\pi} + 1 - \pi$$

.

c. [2 points] Use your formula for $L(x)$ to approximate $e^{\sin\sqrt{38}}$.

Solution:

$$e^{\sin\sqrt{38}} = f(38) \approx L(38) = \frac{38}{4\pi} + 1 - \pi \approx 0.8824.$$

d. [4 points] Recall that the error, $E(x)$, is the actual value of the function minus the value approximated by the tangent line. Given the fact that in this case $E(39) \approx 0.000613$ and $E(40) \approx 0.000719$, would you expect $f''(4\pi^2)$ to be positive or negative? Explain, without doing any calculations.

Solution: The errors are positive, which means that near P the tangent line lies below the curve, so the function is probably concave up at P. Since concave up corresponds to positive second derivative, we should expect the sign of $f''(4\pi^2)$ to be positive.

4. [11 points]

a. [4 points] Find the tangent line approximation of the function

$$p(x) = 1 + x^k$$

near $x = 1$, where k is a positive constant.

Solution:

$$L(x) = k(x - 1) + 2$$

b. [2 points] Suppose you want to use your tangent line from (a) to approximate the number $1 + \sqrt{0.95}$. What values of k and x would you plug in to your answer from (a)?

Solution: We'd take $k = \frac{1}{2}$ and $x = .95$.

c. [2 points] Approximate $1 + \sqrt{0.95}$ using your tangent line from (a).

Solution: We have

$$1 + \sqrt{.95} \approx .5(-.05) + 2 = 1.975.$$

d. [3 points] Determine whether your approximation in (c) is an over- or underestimate. Be sure your reasoning is clear.

Solution: The graph of $1 + x^{.5}$ is just the graph of the square root function shifted up by one, so it's concave down everywhere. It follows that the linear approximation is an overestimate.

10. [10 points] Let $f(x)$ be a function with $f(1) = 5$, $f'(1) = -2$, and $f''(1) = 3$.

a. [2 points] Use the local linearization of $f(x)$ at $x = 1$ to estimate $f(0.9)$.

Solution: The local linearization of $f(x)$ at $x = 1$ is $5 - 2(x - 1)$. Plugging in $x = 0.9$ yields $5 - 2(0.9 - 1) = 5 + 0.2 = 5.2$.

Answer: $f(0.9) \approx$ 5.2

b. [2 points] Do you expect your estimate from Part (a) to be an overestimate or underestimate? To receive any credit on this question, you must justify your answer.

Solution: Since $f''(1) = 3$, the graph of $y = f(x)$ is concave up near $x = 1$. Therefore, the tangent line at $x = 1$ lies under the graph of $f(x)$ near $x = 1$, so we expect this to be an underestimate.

c. [2 points] Use the tangent line approximation of $f'(x)$ near $x = 1$ to estimate $f'(1.1)$.

Solution: The tangent line to $f'(x)$ at $x = 1$ passes through the point $(1, f'(1))$ and has slope $f''(1)$ (as the slope of the derivative function is given by the <u>second</u> derivative). Therefore, the tangent line to $f'(x)$ at $x = 1$ is given by the equation

$$L = -2 + 3(x - 1).$$

Plugging in $x = 1.1$ yields $-2 + 3(1.1 - 1) = -1.7$.

Answer: $f'(1.1) \approx$ -1.7

d. [4 points] Suppose that the tangent line approximation of $f(x)$ near $x = 8$ estimates $f(8.05)$ to be 3.75 and $f(8.1)$ to be 3.6. Find $f(8)$ and $f'(8)$.

Solution: Since the tangent line passes through the points $(8.05, 3.75)$ and $(8.1, 3.6)$, it has slope

$$\frac{3.6 - 3.75}{8.1 - 8.05} = \frac{-0.15}{0.05} = -3.$$

Hence $f'(8) = -3$. Moreover, an equation for this tangent line is therefore

$$L = 3.75 - 3(x - 8.05),$$

so plugging in $x = 8$, it passes through the point $(8, 3.9)$.
By definition of the tangent line, then, we have that $f(x)$ also passes through the point $(8, 3.9)$ and also has slope -3, so we conclude that $f(8) = 3.9$ and $f'(8) = -3$.

Answer: $f(8) =$ 3.9 and $f'(8) =$ -3

1. [5 points] Let $h(x)$ be a differentiable function such that $h'(x)$ is also differentiable everywhere. Suppose that $h(3) = 9$, $h'(3) = 2$, and $h''(x) > 0$ for all real numbers x.

a. [2 points] Let $L(x)$ be the local linearization of $h(x)$ at $x = 3$. Find a formula for $L(x)$.

Solution: The graph of $L(x)$ is the tangent line to the graph of $y = h(x)$ at $x = 3$. This is a line of slope 2 passing through the point $(3, 9)$. So $L(x) = 9 + 2(x - 3)$.

Answer: $L(x) =$ $9 + 2(x - 3)$

b. [3 points] Which of the following equalities could be true?
Circle all the statements that could be true or circle NONE OF THESE.
You do not need to explain your reasoning.

Solution: Since $h''(x) > 0$ for all x, the graph of $h(x)$ is concave up so lies above the graph of $L(x)$. Therefore, $h(-1) > L(-1) = 9 + 2(-4) = 1$.

$h(-1) = -1$

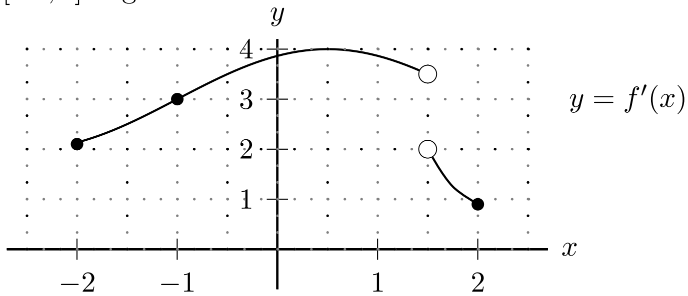
$h(-1) = 0$

$h(-1) = 1$

$\boxed{h(-1) = 2}$

NONE OF THESE

3. [8 points] Suppose $f(x)$ is a function that is continuous on the interval $[-2, 2]$. The graph of $f'(x)$ on the interval $[-2, 2]$ is given below.



a. [3 points] Let $L(x)$ be the local linearization of $f(x)$ at $x = -1$. Using the fact that $f(-1) = 4$, write a formula for $L(x)$.

Solution: $f(-1) = 4$ and $f'(-1) = 3$, so $L(x) = 4 + 3(x - (-1)) = 4 + 3(x + 1)$.

Answer: $L(x) =$ $4 + 3(x + 1)$ **or** $3x + 7$

b. [2 points] Use your formula for $L(x)$ to approximate $f(-0.5)$.

Solution: Since -0.5 is close to -1 we have

$$f(-0.5) \approx L(-0.5) = 4 + 3(-0.5 + 1) = 43(0.5) = 5.5.$$

Answer: $f(-0.5) \approx$ 5.5

c. [3 points] Is your answer from part (b) an overestimate or an underestimate of the actual value of $f(-0.5)$? Justify your answer.

Circle one: overestimate **[underestimate]** CANNOT BE DETERMINED

Justification:

Solution: The function $f'(x)$ is increasing between -2 and 0 so $f(x)$ is concave up over this interval. Therefore the tangent line to the graph of $f(x)$ at $x = -1$ lies below the graph of $f(x)$ between $x = -2$ and $x = 0$. In particular, the local linearization $L(x)$ of $f(x)$ at $x = -1$ gives an underestimate of f on that interval.

10. [9 points] Consider the function h defined by $$h(x) = \begin{cases} Ax^4 & \text{if } x < 2 \\ Bx^3 + 80\ln\left(\frac{x}{2}\right) & \text{if } x \geq 2 \end{cases}$$
where A and B are constants.

a. [6 points] Find values of A and B so that h is differentiable.
Remember to show your work clearly.

Solution: If h is differentiable, it must be continuous, so, in particular,

$$\begin{aligned} \lim_{x\to 2^-} h(x) &= \lim_{x\to 2^+} h(x) \\ A(2)^4 &= B(2)^3 + 80\ln(2/2) \\ 16A &= 8B \\ 2A &= B. \end{aligned}$$

Note that $\frac{d}{dx}\left(Ax^4\right) = 4Ax^3$ and $\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right) = 3Bx^2 + 80(\frac{2}{x})(\frac{1}{2}) = 3Bx^2 + \frac{80}{x}$.
and that both Ax^4 and $Bx^3 + 80\ln\left(\frac{x}{2}\right))$ are differentiable at $x = 2$.
In order for $h(x)$ to be differentiable at $x = 2$, $h'(x)$ must exist at $x = 2$. In particular,

$$\begin{aligned} \lim_{k\to 0^-} \frac{h(2+k) - h(2)}{k} &= \lim_{k\to 0^+} \frac{h(2+k) - h(2)}{k} \\ \left(\frac{d}{dx}(Ax^4)\right)\Bigg|_{x=2} &= \left(\frac{d}{dx}\left(Bx^3 + 80\ln\left(\frac{x}{2}\right)\right)\right)\Bigg|_{x=2} \\ (4Ax^3)\Big|_{x=2} &= \left(3Bx^2 + \frac{80}{x}\right)\Bigg|_{x=2} \quad \text{(i.e. derivatives of the two pieces are equal at } x = 2\text{)} \\ 32A &= 12B + 40. \end{aligned}$$

Since $B = 2A$, we therefore find that

$$\begin{aligned} 32A &= 24A + 40 \\ 8A &= 40 \\ A &= 5 \end{aligned}$$

and hence $B = 2A = 2(5) = 10$.

Answer: $A =$ 5 and $B =$ 10

b. [3 points] Using the values of A and B you found in part **a.**, find the tangent line approximation for $h(x)$ near $x = 1$.

Solution: First, notice that
$$h(1) = 5(1)^4 = 5$$
and
$$h'(1) = 4(5)(1)^3 = 20.$$
So the tangent line approximation for $h(x)$ near $x = 1$ is $y = 5 + 20(x - 1) = 20x - 15$.

Answer: The tangent line approximation is given by $y =$ $5 + 20(x-1)$ **(or** $20x - 15$**)**