Math 115 — Practice for Exam 2

Generated February 20, 2017

Name: **SOLUTIONS** ______________________________

Instructor: ______________________________ Section Number: ____________

1. This exam has 6 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2010	2	5	hyperbola	13	
Fall 2010	2	5		12	
Fall 2011	2	8	rose curve	12	
Winter 2012	2	3		12	
Fall 2012	2	1		12	
Winter 2013	2	7		10	
Total				71	

Recommended time (based on points): 64 minutes

5. [13 points] The equation below implicitly defines a hyperbola.

$$x^2 - y^2 = 2x + xy + y + 2.$$

a. [5 points] Find $\frac{dy}{dx}$.

Solution: We use implicit differentiation:

$$2x - 2y\frac{dy}{dx} = 2 + \left(1 \cdot y + x \cdot \frac{dy}{dx}\right) + \frac{dy}{dx} + 0.$$

Then solve for $\frac{dy}{dx}$:

$$\begin{aligned} 2x - 2 - y &= \frac{dy}{dx}(x + 2y + 1) \\ \frac{dy}{dx} &= \frac{2x - 2 - y}{x + 2y + 1} \end{aligned}$$

b. [4 points] Consider the two points $(4, 2)$ and $(2, -1)$. Show that one of these points lies on the hyperbola defined above, and one does not.

Solution: For the point $(4, 2)$, $x^2 - y^2 = 4^2 - 2^2 = 12$ and $2x + xy + y + 2 = 2(4) + 4(2) + 2 + 2 = 20$ are not equal, so $(4, 2)$ IS NOT on the hyperbola.

For the point $(2, -1)$, $x^2 - y^2 = 2^2 - (-1)^2 = 3$ and $2x + xy + y + 2 = 2(2) + 2(-1) - 1 + 2 = 3$ are equal, so $(2, -1)$ IS on the hyperbola.

c. [4 points] For the point in part (b) which lies on the hyperbola, find the equation of the line which is tangent to the hyperbola at this point.

Solution: From part (a),

$$\frac{dy}{dx} = \frac{2x - 2 - y}{x + 2y + 1},$$

so

$$\frac{dy}{dx}\Big|_{(x,y)=(2,-1)} = \frac{2(2) - 2 - (-1)}{2 + 2(-1) + 1} = \frac{3}{1} = 3.$$

Then the equation of the tangent line is $y = 3(x - 2) - 1$ or $y = 3x - 7$.

5. [12 points] Suppose a curve in the plane is given by the equation

$$\sin(\pi xy) = y - 1.$$

a. [3 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

Solution: At $(1, 1)$, the right hand side is $\sin(\pi) = 0$ and the left hand side is $1 - 1 = 0$. Therefore the point is on the curve since the right and left hand sides are equal.

b. [5 points] Calculate $\frac{dy}{dx}$.

Solution: Taking the derivative with respect to x of the equation, we have

$$\pi\cos(\pi xy) \cdot (y + x\frac{dy}{dx}) = \frac{dy}{dx}.$$

Solving for $\frac{dy}{dx}$, we get

$$\frac{dy}{dx} = \frac{\pi y\cos(\pi xy)}{1 - \pi x\cos(\pi xy)}.$$

c. [4 points] Find the equation for the tangent line to the curve at the point $(1, 1)$.

Solution: The slope of the tangent line to the curve is

$$\frac{dy}{dx}(1, 1) = \frac{\pi\cos(\pi)}{1 - \pi\cos(\pi)} = \frac{-\pi}{1 + \pi}.$$

The equation for the tangent line is

$$y - 1 = \frac{-\pi}{1 + \pi}(x - 1).$$

8. [12 points] The equation $(x^2 + y^2)^2 = 4x^2y$ describes a two-petaled rose curve.

a. [2 points] Verify that the point $(x, y) = (1, 1)$ is on the curve.

Solution: At the point $(x, y) = (1, 1)$,

$$(x^2 + y^2)^2 = (1^2 + 1^2)^2 = 4 = 4(1)^2(1) = 4x^2y.$$

b. [7 points] Calculate dy/dx at $(x, y) = (1, 1)$.

Solution: Differentiating both sides of the equation for the curve with respect to x we have

$$2(x^2 + y^2)\left(2x + 2y\frac{dy}{dx}\right) = 4\left(2xy + x^2\frac{dy}{dx}\right).$$

At the point $(x, y) = (1, 1)$ this equation becomes

$$2(1^2 + 1^2)\left(2(1) + 2(1)\frac{dy}{dx}\right) = 4\left(2(1)(1) + (1)^2\frac{dy}{dx}\right).$$

Simplifying, we have $4(2 + 2\frac{dy}{dx}) = 8 + 4\frac{dy}{dx}$. This gives us that $\frac{dy}{dx} = 0$ at $(x, y) = (1, 1)$.

c. [3 points] Find the equation of the tangent line to the rose curve at the point $(x, y) = (1, 1)$.

Solution: Using point slope form, the tangent line is $y - 1 = 0(x - 1)$. Simplifying, we have that the tangent line to the rose curve at $(x, y) = (1, 1)$ is $y = 1$.

3. [12 points] The following questions relate to the implicit function

$$y^2 + 4x = 4xy^2.$$

a. [4 points] Compute $\dfrac{dy}{dx}$.

Solution: Differentiating the equation with respect to x, we have

$$2y\frac{dy}{dx} + 4 = 4y^2 + 8xy\frac{dy}{dx}.$$

Gathering terms involving $\frac{dy}{dx}$ to one side, the equation becomes

$$2y\frac{dy}{dx} - 8xy\frac{dy}{dx} = 4y^2 - 4$$

which gives the solution

$$\frac{dy}{dx} = \frac{4y^2 - 4}{2y - 8xy}.$$

b. [4 points] Find the equation for the tangent line to this curve at the point $(\frac{1}{3}, 2)$.

Solution: The slope is

$$\left.\frac{dy}{dx}\right|_{(\frac{1}{3},2)} = \frac{4 \cdot 2^2 - 4}{2 \cdot 2 - 8 \cdot \frac{1}{3} \cdot 2} = -9,$$

so by the point-slope formula, the equation is

$$y = -9x + 5.$$

c. [4 points] Find the x- and y-coordinates of all points at which the tangent line to this curve is vertical.

Solution: The slope is undefined as these points, so we must have $2y - 8xy = 0$. Factoring out a $2y$ we get

$$2y(1 - 4x) = 0$$

which gives the solutions $y = 0$ or $x = \frac{1}{4}$. Plugging into the equation for the implicit function, $y = 0$ gives the point $(0, 0)$. However, when we plug in $x = \frac{1}{4}$, we get the equation $y^2 + 1 = y^2$, which has no solutions. Therefore, $(0, 0)$ is the only point at which the tangent line is vertical.

1. [12 points] The following questions relate to the implicit curve $2x^2 + 4x - x^2y^2 + 3y^4 = -1$.

a. [6 points] Calculate $\frac{dy}{dx}$.

Solution: Differentiating both sides with respect to x, we get

$$4x + 4 - 2xy^2 - 2x^2y\,\frac{dy}{dx} + 12y^3\,\frac{dy}{dx} = 0.$$

Moving all terms with no $\frac{dy}{dx}$ to the other side and factoring out $\frac{dy}{dx}$ gives us

$$\frac{dy}{dx}(12y^3 - 2x^2y) = 2xy^2 - 4x - 4.$$

So

$$\frac{dy}{dx} = \frac{2xy^2 - 4x - 4}{12y^3 - 2x^2y} = \frac{xy^2 - 2x - 2}{6y^3 - x^2y}.$$

b. [2 points] Q is the only point on the curve that has a y-coordinate of 1. Find the x-coordinate of Q.

Solution: Plugging $y = 1$ into the equation for the curve gives us

$$2x^2 + 4x - x^2 + 3 = -1.$$

Moving all the terms to the left, we get

$$x^2 + 4x + 4 = 0.$$

This factors as $(x + 2)^2 = 0$, so $x = -2$.

c. [4 points] Find the equation of the tangent line to the curve at Q.

Solution: To find the slope, we plug in $x = -2$ and $y = 1$ to $\frac{dy}{dx}$.

$$\text{slope} = \frac{-2 + 4 - 2}{6 - 4} = 0.$$

Thus, the tangent line is the horizontal line passing through Q, which has equation $y = 1$.

7. [10 points] For each real number k, there is a curve in the plane given by the equation

$$e^{y^2} = x^3 + k.$$

a. [4 points] Find $\frac{dy}{dx}$.

Solution: We have

$$2ye^{y^2}\frac{dy}{dx} = 3x^2,$$

so

$$\frac{dy}{dx} = \frac{3x^2}{2ye^{y^2}}$$

b. [3 points] Suppose that $k = 9$. There are two points on the curve where the tangent line is horizontal. Find the x and y coordinates of each one.

Solution: Horizontal tangent lines occur when the numerator of the derivative is zero, so in this case $x = 0$. To solve for the y-coordinate, we have

$$e^{y^2} = 9$$

so $y = \pm\sqrt{\ln(9)}$.

c. [3 points] Now suppose that $k = \frac{1}{2}$. How many points are there where the curve has a horizontal tangent line?

Solution: Again we get $x = 0$. Now if we try to solve for y we have

$$y^2 = \ln\left(\frac{1}{2}\right) < 0$$

and so there are no points where the curve has a horizontal tangent line.