4. [13 points] One of the ways Captain Christina likes to relax in her retirement is to go for long walks around her neighborhood. She has noticed that early every Tuesday morning, a truck delivers butter to a local bakery famous for its cookie dough. Consider the following functions:

- Let $C(b)$ be the bakery's cost, in dollars, to buy b pounds of butter.
- Let $K(b)$ be the amount of cookie dough, in cups, the bakery makes from b pounds of butter.
- Let $u(t)$ be the instantaneous rate, in pounds per hour, at which butter is being unloaded t hours after 4 am.

Assume that C, K, and u are invertible and differentiable.

a. [2 points] Interpret $K(C^{-1}(10)) = 20$ in the context of this problem.
Use a complete sentence and include units.

Solution: If the bakery spends \$10 on butter, then it can make 20 cups of cookie dough.

b. [3 points] Interpret $\int_5^{12} K'(b)\, db = 40$ in the context of this problem.
Use a complete sentence and include units.

Solution: 12 pounds of butter makes 40 cups more cookie dough than 5 pounds of butter does.

c. [2 points] Give a single mathematical equality involving the derivative of C which supports the following claim:
It costs the bakery approximately \$0.70 less to buy 14.8 pounds of butter than to buy 15 pounds of butter.

Answer: $C'(15) = 3.5$

d. [3 points] Give a single mathematical equality which expresses the following claim:
The number of pounds of butter unloaded between 5 and 8 am is twice as many as the bakery needs to make 5000 cups of cookie dough.

Answer: $\int_1^4 u(t)\, dt = 2K^{-1}(5000)$

e. [3 points] Assume that $u(t) > 0$ and $u'(t) < 0$ for $0 \le t \le 4$ and that $u(2) = 800$.
Rank the following quantities in order from least to greatest by filling in the blanks below with the options I-IV.

I. 0 II. 800 III. $\int_1^2 u(t)\, dt$ IV. $\int_2^3 u(t)\, dt$

Solution: Since $u(t) > 0$, both integrals are greater than 0. Since $u'(t) < 0$, $u(t)$ is a decreasing function. Estimating $\int_1^2 u(t)\, dt$ with a right sum with one subdivision yields an underestimate of 800, and likewise, estimating $\int_2^3 u(t)\, dt$ with a left sum with one subdivision yields an overestimate of 800.

$0 < \int_2^3 u(t)\, dt < 800 < \int_1^2 u(t)\, dt$

8. [11 points] The energy, in megajoules (MJ), produced by a wind turbine depends on the speed of the wind. In particular, suppose $P(s)$ is the power, in megajoules per hour (MJ/h), produced by the turbine when the speed of the wind is s kilometers per hour (km/h). Also suppose that $W(t)$ gives the wind speed, in km/h, at the turbine's location t hours after noon on a typical day.

Assume that $P(s)$ is invertible, and that both $P(s)$ and $W(t)$ are differentiable.

a. [2 points] Give a practical interpretation of the equation $P(W(0)) = 8$.

Solution: At noon on a typical day, the turbine produces 8 MJ/h of power.

b. [3 points] Give a practical interpretation of the equation $\int_0^5 P(W(t))\,dt = 46$.

Solution: From noon to 5 p.m. on a typical day, the turbine generates 46 MJ of energy.

c. [3 points] Complete the following sentence to give a practical interpretation of the equation

$$W'(4) = 21$$

From 4 pm to 4:10 pm, ...

Solution: the wind speed at the turbine's location increases by approximately 3.5 km/h.

d. [3 points] Circle the one statement below that is best supported by the equation

$$(P^{-1})'(13) = 2.9.$$

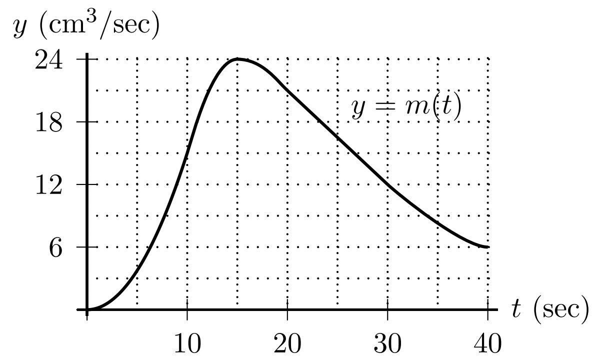
i. *If the turbine is producing 13 MJ/h of power, the wind speed must increase by approximately 2.9 km/h to produce an additional MJ/h of power.*

ii. If the wind is blowing at 13 km/h and increases to 14 km/h, the power produced by the turbine will increase by about 2.9 MJ/h.

iii. If the wind speed is 13 km/h, the power generation of the turbine will increase by one MJ/h if the wind speed increases to about 15.9 km/h.

iv. When the turbine is generating 13 MJ/h of power, an increase of one km/h in wind speed will produce approximately 2.9 MJ/h more power.

2. [10 points]
Kathy puts a very large marshmallow in the microwave for forty seconds and watches as it inflates. Let $m(t)$ be the rate of change of the volume of the marshmallow, in $\mathrm{cm}^3/\mathrm{sec}$, t seconds after Kathy puts it in the microwave. The graph of $y = m(t)$ is shown to the right.



a. [2 points] Write a definite integral equal to the total change in volume, in cm^3, of the marshmallow while in the microwave. (You do not need to evaluate the integral.)

Answer: $\int_0^{40} m(t)\,dt$

b. [3 points] Estimate your integral from part (a) using a right-hand sum with $\Delta t = 10$. Be sure to write out all of the terms in the sum.

Solution: A right-hand sum from 0 to 40 with $\Delta t = 10$ will involve the values at $t = 10$, 20, 30, and 40:

$$10m(10) + 10m(20) + 10m(30) + 10m(40) = 10(15 + 21 + 12 + 6) = 540.$$

Since $m(t)$ has units of cm^3/sec and t has units of sec, the integral has units of cm^3, which agrees with it being a change in volume.

Answer: 540 $\mathbf{cm^3}$

c. [5 points] Assume that throughout its time in the microwave, the marshmallow is a cylinder. After 30 seconds in the microwave, the marshmallow is a cylinder with radius 4.5 cm and height 11 cm. At that moment, the height is increasing at 0.08 cm/sec. How fast is the radius of the marshmallow increasing at that moment?
Recall that the volume V of a cylinder of radius r and height h is $V = \pi r^2 h$, and remember to include units.

Solution: Differentiating both sides of the volume equation with respect to t yields

$$\frac{dV}{dt} = 2\pi r h \frac{dr}{dt} + \pi r^2 \frac{dh}{dt}.$$

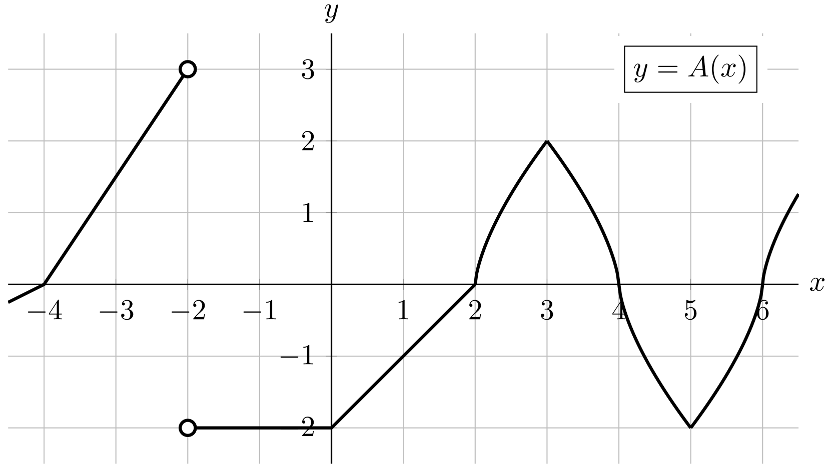
We are told that at this moment, $r = 4.5$, $h = 11$, and $\frac{dh}{dt} = 0.08$. Further, since $m(t) = \frac{dV}{dt}$, we can read from the graph above that at $t = 30$, we have $\frac{dV}{dt} = 12$. Plugging these in, we have

$$12 = 99\pi \frac{dr}{dt} + 1.62\pi,$$

so solving for $\frac{dr}{dt}$ yields a rate of about 0.022 cm/sec.

Answer: 0.022 **cm/sec**

3. [10 points] A portion of the graph of a function $A(x)$ is shown below. Note that the part of the graph on the interval $[4, 6]$ can be obtained from the part of the graph on the interval $[2, 4]$ by shifting it two units to the right and reflecting it over the x-axis.



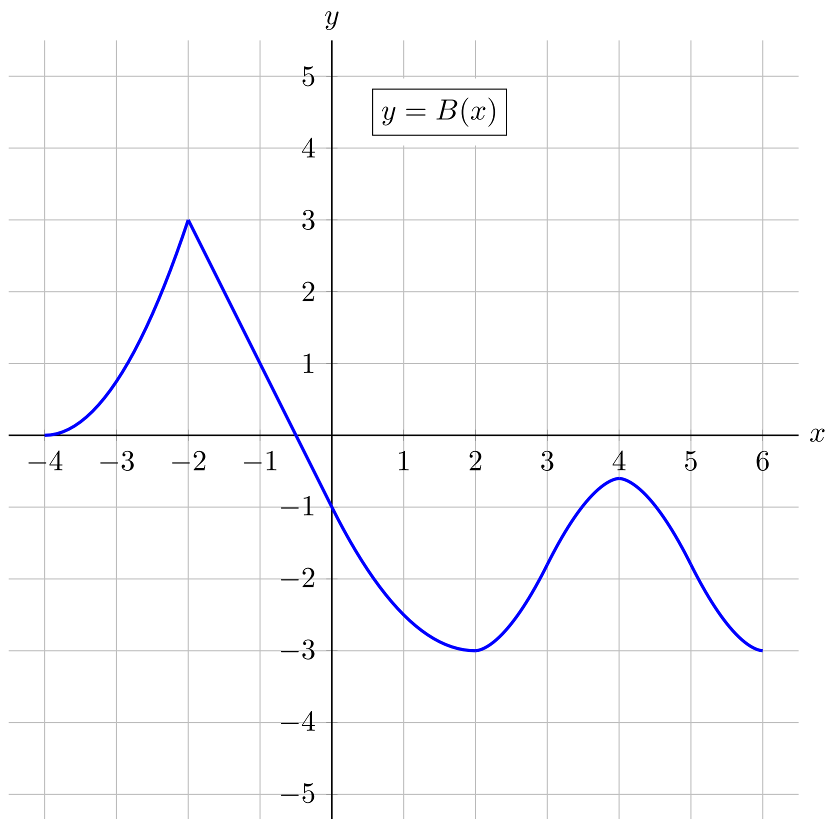
Let $B(x)$ be the continuous antiderivative of $A(x)$ passing through the point $(-1, 1)$.

a. [5 points] Use the graph above to complete the table below with the **exact** values of $B(x)$.

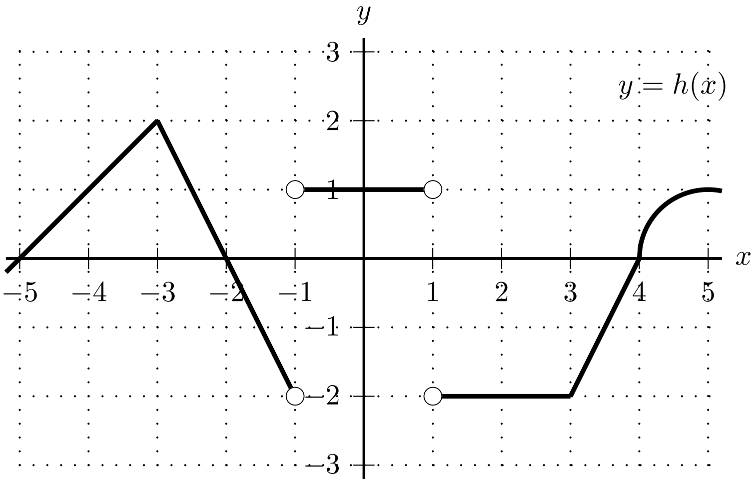
x	-4	-2	-1	0	2	6
$B(x)$	0	3	1	-1	-3	-3

b. [5 points] On the axes below, sketch a detailed graph of $y = B(x)$ for $-4 \leq x \leq 6$. Be sure that you pay close attention to each of the following:

- where $B(x)$ is and is not differentiable,
- the values of $B(x)$ you found in the table above and at local extrema of B,
- where $B(x)$ is increasing/decreasing/constant, and the concavity of $B(x)$.



5. [9 points] The graph of a portion of $y = h(x)$ is shown below.



Note: The portion of the graph of $h(x)$ between $x = 4$ and $x = 5$ is part of a circle of radius 1 centered at the point $(5, 0)$.

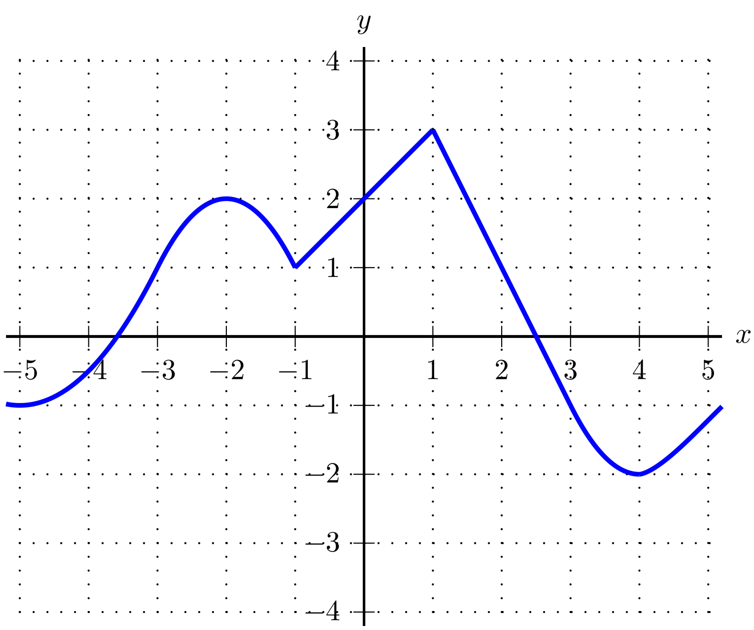
Let $H(x)$ be the <u>continuous</u> antiderivative of $h(x)$ with $H(0) = 2$.

a. Complete the following table with the <u>exact</u> values of $H(x)$.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
$H(x)$	-1	-0.5	1	2	1	2	3	1	-1	-2	$-2+\frac{\pi}{4}$

b. On the axes below, sketch the graph of $y = H(x)$. Be sure that you pay close attention to each of the following:

- where $H(x)$ is and is not differentiable
- the values of $H(x)$ from the table above
- the sign of $H(x)$, where $H(x)$ is increasing/decreasing/constant, and the concavity of $H(x)$



8. [7 points] Mr. R. DeVark discovers that there is a loud humming sound emanating from a tree in his backyard. The volume of the sound at any point in the yard is a function of the point's distance from the tree.

- Let $V(x)$ be the rate of change (in decibels per meter) of the volume of the sound where x is the distance (in meters) from the tree.
- Let $K(t)$ be the distance, in meters, of Mr. DeVark from the tree t seconds after he first notices the sound.

Assume that K is invertible and that V, K, and K^{-1} are differentiable.

a. [3 points] Give a practical interpretation of the equation $\displaystyle\int_{10}^{40} V(x)\,dx = -5$ in the context of this problem. *Remember to use a complete sentence and include units.*

Solution: The humming sound is 5 decibels louder at a point 10 meters away from the tree than it is at a point 40 meters from the tree.

b. [2 points] Which one of the following expressions represents the instantaneous rate of change (in decibels per second) of the volume at which Mr. DeVark hears the sound 30 seconds after he first notices the sound? Circle the one best answer.

$V(30)$	$V(30)K(30)$	$V'(30)K(30)$
$V'(30)$	$V(30)K'(30)$	$V'(30)K'(30)$
$V(K(30))$	$V(K(30))K(30)$	$V'(K(30))K(30)$
$V(K'(30))$	$\boxed{V(K(30))K'(30)}$	$V'(K(30))K'(30)$
$V'(K(30))$		
$V'(K'(30))$	$V(K'(30))K'(30)$	$V'(K'(30))K'(30)$

c. [2 points] Which of the following is the best interpretation of the equation $(K^{-1})'(15) = -2$? Circle the one best answer.

i. Between 15 and 15.5 seconds after Mr. DeVark notices the humming sound, he moves about 1 meter closer to the tree.

ii. It takes about 1 second for Mr. DeVark to go from being 15 meters away from the tree to 14.5 meters away from the tree.

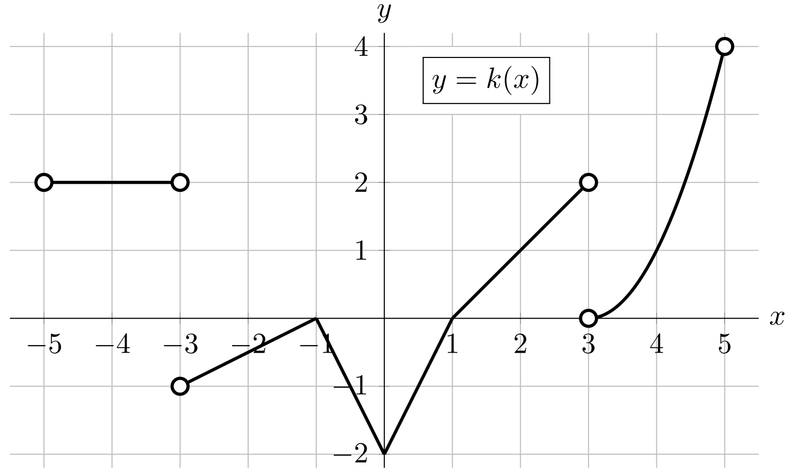
iii. The volume of the humming sound is about 1 decibel lower at a point 15.5 meters from the tree than it is at a point 15 meters from the tree.

iv. When Mr. DeVark is 15 meters away from the tree, it is about 2 seconds before he notices the humming sound

v. The volume of the humming sound Mr. DeVark hears is about 1 decibel lower 15 seconds after he first notices it than 0.5 seconds later.

vi. When Mr. DeVark is 15 meters away from the tree, he moves about 2 meters closer to the tree in the next second.

11. [10 points] The graph of a portion of $y = k(x)$ is shown below. Note that for $3 < x < 5$, the graph of $k(x)$ is a portion of the graph obtained by shifting $y = x^2$ three units to the right.



Let $K(x)$ be the continuous antiderivative of $k(x)$ passing through the point $(-1, 1)$.

a. [5 points] Use the graph to complete the table below with the exact values of $K(x)$.

x	-5	-3	-1	1	3	5
$K(x)$	-2	2	1	-1	1	$\frac{11}{3}$

b. [5 points] On the axes below, sketch a detailed graph of $y = K(x)$ for $-5 < x < 5$. Be sure that you pay close attention to each of the following:

- where $K(x)$ is and is not differentiable,
- the values of $K(x)$ you found in the table above,
- where $K(x)$ is increasing/decreasing/constant, and the concavity of $K(x)$.