2. [10 points] Jane has a company that produces a protein powder for an energy shake. The cost, in dollars, of producing m pounds of protein powder is given by the function

$$C(m) = \begin{cases} \frac{1}{4}(m+2)^2 + 8 & 0 \leq m < 16 \\ 2m + 57 & 16 \leq m \leq 30. \end{cases}$$

The revenue, in dollars, of selling m pounds of protein powder is given by $R(m) = 5m$.

a. [1 point] What is the price, in dollars, at which Jane sells each pound of the protein powder?

Solution: **Answer=** 5

b. [1 point] What is the fixed cost, in dollars, of producing Jane's protein powder?

Solution: Fixed cost is $C(0) = \frac{1}{4}(0+2)^2 + 8 = 9$. **Answer=** 9

c. [2 points] Find all values of $16 \leq m \leq 30$ for which Jane's profit is positive.

Solution: Jane breaks even when $R(m) = C(m)$. That occurs on $16 \leq m \leq 30$ when $2m + 57 = 5m$. Then Jane breaks even if $m = 19$. We can see that the profit of selling 20 pounds ($m = 20$) is $5(20) - (2(20) + 57) = 3 > 0$. Since both $R(m)$ and $C(m)$ are continuous on $[16, 30]$ then Jane's profit is positive for $19 < m \leq 30$.

Answer: $19 < m \leq 30$

d. [2 points] Find all the values of $0 \leq m \leq 30$ where the marginal cost is equal to the marginal revenue for the protein powder. Show all your work to justify your answer.

Solution: Note that

$$MC(m) = \begin{cases} \frac{1}{2}(m+2) & 0 < m < 16 \\ 2 & 16 < m < 30. \end{cases}$$

and $MR(m) = 5$. Hence $MC = MR$ if $\frac{1}{2}(m+2) = 5$ on $0 < m < 16$. Solving for m we get $m = 8$. We do not consider the interval $16 < m < 30$ since in this case, there is no m that yields $MC = MR$.

Answer= 8

e. [4 points] What is the maximum profit that Jane can make if she sells at most 30 pounds of protein powder? Use calculus to find and justify your answer, and make sure to provide enough evidence to fully justify your answer.

Solution: To find the global maximum of the profit $P(m)$ in dollars of selling m pounds of protein powder, we first need to find its critical points on $0 \leq m \leq 30$. Critical points satisfy either $MC = MR$ ($P'(m) = 0$) or $P'(m)$ does not exist. Hence $m = 8$ is a critical point. The value $m = 16$ is a critical point of $P(m) = R(m) - C(m)$ since $C(m)$ is not differentiable at $m = 16$.

m	0	8	16	30
$P(m) = R(m) - C(m)$	-5	7	-9	33

Answer: 33 dollars.

3. [12 points] Oren plans to grow kale on his community garden plot, and he has determined that he can grow up to 160 bunches of kale on his plot. Oren can sell the first 100 bunches at the market and any remaining bunches to wholesalers. The revenue in dollars that Oren will take in from selling b bunches of kale is given by

$$R(b) = \begin{cases} 6b & \text{for } 0 \leq b \leq 100 \\ 4b + 200 & \text{for } 100 < b \leq 160. \end{cases}$$

a. [2 points] Use the formula above to answer each of the following questions.

i. What is the price (in dollars) that Oren will charge for each bunch of kale he sells at the market?

Answer: \$6

ii. What is the price (in dollars) that Oren will charge for each bunch of kale he sells to wholesalers?

Answer: \$4

For $0 \leq b \leq 160$, it will cost Oren $C(b) = 20 + 3b + 24\sqrt{b}$ dollars to grow b bunches of kale.

b. [1 point] What is the fixed cost (in dollars) of Oren's kale growing operation?

Answer: \$20

c. [4 points] At what production level(s) does Oren's marginal revenue equal his marginal cost?

Solution: Oren's marginal revenue is $R'(b) = 6$ for $0 < b < 100$ and $R'(b) = 4$ for $100 < b < 160$. His marginal cost is $C'(b) = 3 + 12/\sqrt{b}$.
Thus, $R'(b) = C'(b)$ for $b = 16$ and $b = 144$.

Answer: **at 16 bunches and 144 bunches**

d. [5 points] Assuming Oren can grow up to 160 bunches of kale, how many bunches of kale should he grow in order to maximize his profit, and what is the maximum possible profit? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: Since Oren's profit function, $\pi(b) = R(b) - C(b)$, is continuous on $0 \leq b \leq 160$, it has a global maximum (by the Extreme Value Theorem) and the global maximum occurs at a critical point or an endpoint.
The critical points of $\pi(b)$ occur when $\pi'(b) = 0$ (at $b = 16$ and 144 (when MR=MC)), and when $\pi'(b)$ is undefined (at $b = 100$).
We check the value of $\pi(b)$ at the critical points and end points:
$\pi(0) = -20$, $\pi(16) - -68$, $\pi(100) = 40$, $\pi(144) = 36$, and $\pi(160) \approx 36.42$, and conclude that the maximum occurs at $b = 100$, with a resulting maximum profit of \$40.

Answer: bunches of kale: 100 and max profit: \$40

3. [16 points] Eddie and Laura have signed an exclusive contract to begin producing the world's first caffeinated soup, called Minestromnia. If they charge \$4.00 per liter or more for the soup, then nobody will buy it. Otherwise, if they charge p dollars per liter for the soup, they will sell $g(p)$ liters, where

$$g(p) = 500(16 - p^2).$$

a. [3 points] Write an expression for the revenue $R(p)$ that Eddie and Laura will generate if they charge p dollars per liter of soup.

Solution: The revenue is the price times the number of liters sold, so

$$R(p) = pg(p) = 500p(16 - p^2)$$

(To be really precise, we should say

$$R(p) = \begin{cases} 500p(16 - p^2), & 0 \le p < 4 \\ 0, & p \ge 4 \end{cases}$$

or something like this.)

b. [3 points] The ingredients in a liter of Minestromnia cost \$1.00. To start their business, Eddie and Laura need to purchase a very large soup kettle and other equipment at a total cost of \$700.00. Write an expression for the total cost $C(p)$, including fixed costs, of producing $g(p)$ liters of soup.

Solution: The fixed costs are 700 and each liter costs a dollar, for a total cost of

$$C(p) = 700 + g(p) = 700 + 500(16 - p^2).$$

c. [6 points] What price should Eddie and Laura charge per liter of Minestromnia in order to maximize their profits? Be sure to explain how you know that this price produces the maximum possible profit.

Solution: Write $\pi(p) = R(p) - C(p)$. Then after some algebra, we get $\pi(p) = -500p^3 + 500p^2 + 8000p - 8700$, so

$$\pi'(p) = -1500p^2 + 1000p + 8000$$

Setting $\pi'(p)$ equal to zero gives a quadratic equation, whose solutions are $2\frac{2}{3} \approx 2.67$ and the illogical -2. So the critical point is $p = 2.67$. Plugging in the critical point gives

$$\pi(2.67) \approx 6707.$$

The reasonable prices that Eddie and Laura can set lie in a closed interval: $0 \le p \le 5$. The profit is clearly negative at both endpoints (they aren't getting any revenue at either endpoint) so the maximum profit occurs when the price is approximately \$2.67.

3. (continued)

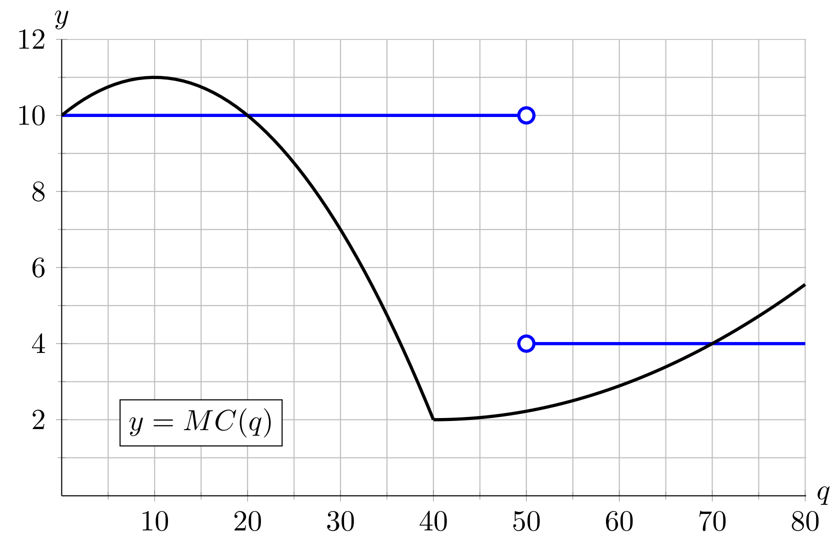
d. [4 points] Give a practical interpretation of the formula

$$g'(3.5) = -3500$$

that begins with
"If Eddie and Laura decrease the price of the soup from \$3.50 per liter to \$3.40 per liter ..."

Solution: ... they expect to sell about 350 more liters of Minestromnia.

5. [10 points] Javier plans to make and sell his own all-natural shampoo. The graph below shows the marginal cost $MC(q)$, in dollars per liter, of q liters of shampoo. In order to start making shampoo, Javier must first spend \$25 on supplies, but he has no other fixed costs.



Javier can sell up to 50 liters of shampoo for \$10 per liter. Any additional shampoo can be sold to a local salon for \$4 per liter. Throughout this problem, you do not need to show work.

a. [2 points] On the axes above, carefully sketch the graph of the marginal revenue $MR(q)$, in dollars per liter, of q liters of shampoo.

Solution: See above.

b. [1 point] At what value(s) of q in the interval $[0, 80]$ is marginal cost maximized?

Answer: 10

c. [1 point] At what value(s) of q in the interval $[0, 80]$ is cost maximized?

Answer: 80

d. [2 points] At which values of q in the interval $[0, 80]$ is profit increasing? Give your answer as one or more intervals.

Answer: $(20, 70)$

e. [1 point] How many liters of shampoo should Javier make in order to maximize his profit?

Answer: 70

5. [8 points] Reggie is starting a fruit punch company. He has determined that the total cost, in dollars, for him to produce q gallons of fruit punch can be modeled by

$$C(q) = 100 + q + 25e^{q/100}.$$

Reggie can sell up to 100 gallons to Chris at a price of \$4 per gallon, and he can sell the rest to Alice at a price of \$3 per gallon. Assume that Reggie sells all of the fruit punch that he produces.

Note: Assume that the quantities of fruit punch produced and sold do not have to be whole numbers of gallons. (For example, Reggie could produce exactly $50\sqrt{2}$ gallons of fruit punch and sell all of these to Chris, who would pay a total of $200\sqrt{2}$ dollars for them.)

a. [4 points] For what quantities of fruit punch sold would Reggie's marginal revenue equal his marginal cost?

Solution: Reggie's marginal cost is $MC = C'(q) = 1 + \frac{1}{4}e^{q/100}$ and his marginal revenue is $MR = \begin{cases} 4 & \text{if } 0 < q < 100 \\ 3 & \text{if } 100 < q. \end{cases}$

So we solve $MR = MC$ separately for the two intervals $0 < q < 100$ and $q > 100$.

For $0 < q < 100$:

$$\begin{aligned} 1 + \frac{1}{4}e^{q/100} &= 4 \\ \frac{1}{4}e^{q/100} &= 3 \\ e^{q/100} &= 12 \\ q = 100\ln(12) &\approx 248.49. \end{aligned}$$

So marginal cost does not equal marginal revenue anywhere on the interval $0 < q < 100$ (because $100\ln(12) > 100$).

For $q > 100$:

$$\begin{aligned} 1 + \frac{1}{4}e^{q/100} &= 3 \\ \frac{1}{4}e^{q/100} &= 2 \\ e^{q/100} &= 8 \\ q = 100\ln(8) &\approx 207.94 \end{aligned}$$

Hence, marginal revenue equals marginal cost at $q = 100\ln(8)$.

Answer: $100\ln(8) \approx 207.94$ **gallons**

b. [4 points] Assuming that Reggie can produce at most 200 gallons of fruit punch, how much fruit punch should he produce in order to maximize his profit, and what would that maximum profit be? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: First, we find all critical points of the profit function $\pi(q)$ in the interval $0 \le q \le 200$. In part **a.**, we found that $\pi'(q) = 0$ only at $q \approx 207.94$, which is not in the interval $[0, 200]$. The other critical points of $\pi(q)$ occur where $\pi'(q)$ is not defined, namely, at $q = 100$.

Note that Reggie's revenue is a continuous function of q. So $\pi(q)$ is continuous on the interval $[0, 200]$ and we can apply the Extreme Value Theorem. It therefore suffices to compare the value of $\pi(q)$ at the endpoints ($q = 0$ and $q = 200$) and at the critical point ($q = 100$):

$$\begin{aligned} \pi(0) &= 0 - (100 + 0 + 25e^0) = -125 \\ \pi(100) &= 4(100) - (100 + 100 + 25e^1) \approx 132.04 \\ \pi(200) &= 4(100) + 3(100) - (100 + 200 + 25e^2) \approx 215.27 \end{aligned}$$

Hence, Reggie should produce 200 gallons of fruit punch for a profit of about \$215.27.

Answer: gallons of fruit punch: 200 and max profit: \$215.27

6. [11 points] Ben has recently acquired a cabbage press and is opening a business selling cabbage juice. Let $R(x)$ and $C(x)$ be the revenue and cost, in dollars, of selling and producing x cups of cabbage juice. Ben only has resources to produce up to a hundred cups. After some research, Ben determines that

$$R(x) = 6x - \frac{1}{40}x^2 \qquad \text{for} \qquad 0 \le x \le 100$$

and

$$C(x) = \begin{cases} 60 + 2x & 0 \le x \le 20 \\ 70 + 1.5x & 20 < x \le 100. \end{cases}$$

a. [3 points] What is the smallest quantity of juice Ben will need to sell in order for his profit to not be negative? Round your answer to the nearest hundredth of a cup. Show your work.

Solution: We consider values of x such that $R(x) = C(x)$. We first look in $[0, 20]$

$$60 + 2x = 6x - \frac{1}{40}x^2 \qquad \text{or} \qquad \frac{1}{40}x^2 - 4x + 60 = 0.$$

Using the quadratic formula we get $x = 80 \pm 20\sqrt{10}$. Only one of these two solutions, $x = 80 - 20\sqrt{10} \approx 16.75$, is in the interval $[0, 20]$.

The last step is to verify that $R(x) - C(x)$ is negative on the interval $[0, \; 80 - 20\sqrt{10})$ and positive on the interval $(80 - 20\sqrt{10}, \; 20]$. We can test this by picking points in each interval. For example, $R(0) - C(0) = -60$ and $R(20) - C(20) = 10$. **Answer:** 16.75 cups.

For the following parts, determine how many cups of cabbage juice Ben needs to sell in order to maximize the given quantity. If there is no such value, write NONE. Use calculus to find and justify your answers.

b. [3 points] Ben's revenue.

Solution: The critical points of $R(x)$ can be found by solving $R'(x) = 6 - \frac{1}{20}x = 0$. This occurs when $x = 120$ which is not in $[0, 100]$. Hence the maximum has to be at one of the endpoints $x = 0$ or $x = 100$. Since $R(0) = 0$ and $R(100) = 350$, the maximum revenue is attained at $x = 100$. **Answer:** 100 cups.

c. [5 points] Ben's profit.

Solution: Since $P(20) = 10$ and

$$\lim_{x \to 20^-} P(x) = \lim_{x \to 20^-} 6x - \frac{1}{40}x^2 - (60 + 2x) = 10$$

and

$$\lim_{x \to 20^+} P(x) = \lim_{x \to 20^+} 6x - \frac{1}{40}x^2 - (70 + 1.5x) = 10.$$

Then $P(x)$ is continuous on $[0, 20]$. The critical points of $P(x)$ can be found by solving $P'(x) = R'(x) - C'(x) = 0$ in the intervals $(0, 20)$ and $(20, 100)$.

- On $(0, 20)$ we need to solve $6 - \dfrac{1}{20}x = 2$. This yields $x = 80$ (outside the interval).
- On $(20, 100)$ we need to solve $6 - \dfrac{1}{20}x = 1.5$. This yields $x = 90$.

Hence the critical points of $P(x)$ are $x = 0$ and $x = 90$. Since $P(x)$ is continuous then the global maximum must lie on the critical points or in the endpoints.

x	0	20	90	100
$P(x)$	-60	10	132.5	130

Answer: 90 cups.

7. [10 points] Zerina owns a small business selling custom screen-printed and embroidered apparel.

a. Zerina receives orders for embroidered polo shirts, which she sells for \$11 each. The cost, in dollars, for her to complete an order of q embroidered polo shirts is

$$C(q) = \begin{cases} 6q - \frac{1}{8}q^2 + \frac{56}{9} & 0 \le q \le 16 \\ \frac{2}{9}q^{3/2} + 10q - 104 & q > 16. \end{cases}$$

Note that $C(q)$ is continuous for all $q \ge 0$.

i. [1 point] What is the fixed cost, in dollars, of an order of embroidered polo shirts?

Answer: 56/9

ii. [5 points] Find the quantity q of embroidered polo shirts in an order that would result in the most profit for Zerina. Assume that, because of storage constraints, Zerina cannot accept an order for more than 80 embroidered polo shirts. Use calculus to find and justify your answer, and make sure you provide enough evidence to fully justify your answer.

Solution: We are given $C(q)$, and know that $R(q) = 11q$. Then since $\pi(q) = R(q) - C(q)$, any point at which $MR = MC$ is a critical point of $\pi(q)$. Now $MR(q) = 11$ and

$$MC(q) = \begin{cases} 6 - \frac{1}{4}q & 0 \le q < 16 \\ \frac{1}{3}q^{1/2} + 10 & q > 16. \end{cases}$$

We set $MR = MC$ in both of these cases:

$$6 - \frac{1}{4}q = 11 \qquad\qquad \frac{1}{3}q^{1/2} + 10 = 11$$

$$-\frac{1}{4}q = 5 \qquad\qquad q^{1/2} = 3$$

$$q = -20 \qquad\qquad q = 9$$

but neither critical point falls within the domain of the appropriate formula. So there are no points at which $MR = MC$. However, MC is undefined at $q = 16$, since if we plug 16 in to both pieces of $MC(q)$ we get different values. Therefore $\pi'(q)$ is also undefined at $q = 16$. This is the only critical point.

So the possible locations for the global maximum are the endpoints 0 and 80 and the critical point 16. Since $\pi(0) = -56/9$, $\pi(80) \approx 25$, and $\pi(16) \approx 42$, an order of 16 polo shirts would result in the most profit for Zerina.

Answer: $q =$ **16**

b. [3 points] Zerina also receives orders for screen-printed t-shirts. When a customer places such an order, they pay a \$6 setup fee, plus \$9 per t-shirt for the first 20 t-shirts ordered. Any additional t-shirts ordered only cost \$7 per t-shirt. Let $P(s)$ be the total price, in dollars, a customer pays for an order of s screen-printed t-shirts. Find a formula for $P(s)$.

Answer: $$P(s) = \begin{cases} 6 + 9s & \text{if } 0 \le s \le 20 \\ 186 + 7(s-20) & \text{if } s > 20 \end{cases}$$

10. [10 points] Yukiko has a small orchard where she grows Michigan apples. After careful study last season, Yukiko found that the total cost, in dollars, of producing a bushels of apples can be modeled by

$$C(a) = -25500 + 26000e^{0.002a}$$

for $0 \le a \le 320$.

Qabil has promised to buy up to 100 bushels of apples for his famous apple ice cream. If Yukiko has any remaining apples, she has an agreement to sell them to Xanthippe's cider mill at a reduced price. Let $R(a)$ be the revenue generated from selling a bushels of apples. Then

$$R(a) = \begin{cases} 70a & \text{if } 0 \le a \le 100 \\ 2000 + 50a & \text{if } 100 < a \le 320. \end{cases}$$

a. [1 point] How much will Xanthippe's cider mill pay per bushel?

Answer: **\$50**

b. [1 point] What is Yukiko's fixed cost?

Answer: **\$500**

c. [4 points] For what quantities of bushels of apples sold would Yukiko's marginal revenue equal her marginal cost? Write NONE if appropriate.

Solution: Yukiko's marginal revenue is given by

$$MR = \begin{cases} 70 & \text{if } 0 < a < 100 \\ 50 & \text{if } 100 < a < 320 \end{cases}$$

and her marginal cost is $52e^{0.002a}$. We have $52e^{0.002a} = 70$ when $a \approx 148.63$, but this is greater than 100, so it is not in the correct domain. Also $52e^{0.002a} = 50$ when $a \approx -19.61$, which is also not in the domain. Thus there are no values of a where $MC = MR$.

Answer: **None**

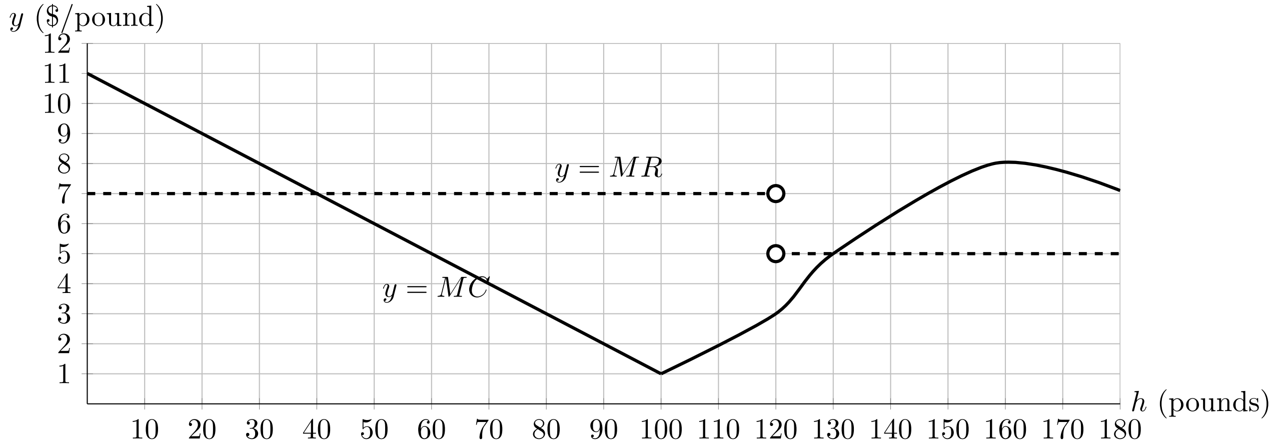
d. [4 points] Assuming Yukiko can produce up to 320 bushels of apples, how many bushels should she produce in order to maximize her profit, and what would that maximum profit be? You must use calculus to find and justify your answer. Make sure to provide enough evidence to justify your answer fully.

Solution: Let $\pi(a) = R(a) - C(a)$ be the profit function. Note that $C(a)$ and $R(a)$ are continuous on this closed interval, so we can apply the Extreme Value Theorem. Since we found in the previous part that MC and MR are never equal, we only need to consider endpoints and points where $\pi'(a)$ does not exist. This happens when $q = 100$. Using the formulas we've been given, we find

$$\pi(0) = -500$$
$$\pi(100) \approx 743.53$$
$$\pi(320) \approx -5808.50$$

Answer: bushels of apples: **100** and max profit: **\$743.53**

10. [10 points] The Happy Hives Bee Farm sells honey. The graph below shows marginal revenue MR (dashed) and marginal cost MC (solid), in dollars per pound, where h is the number of pounds of honey.



a. [7 points] Use the graph to estimate the answers to the following questions. You do not need to show work. If an answer can't be found with the information given, write "NEI".

i) For what value(s) of h in the interval $[0, 180]$ is the cost function C minimized?

Answer: $h = 0$.

ii) For what value(s) of h in the interval $[0, 180]$ is MC minimized?

Answer: $h = 100$.

iii) For what value(s) of h in the interval $[0, 180]$ is profit maximized?

Answer: $h = 130$.

iv) What are the fixed costs of the farm?

Answer: NEI

v) For what values of h in the interval $[0, 180]$ is the profit function concave up?

Answer: $(0, 100) \bigcup (160, 180)$

10. [8 points] Gen is setting up a business selling hot chocolate in Srebmun Foyoj and, due to local restrictions, she will be able to produce and sell no more than 200 gallons. She has determined that the total cost, in dollars, for her to produce g gallons of hot chocolate can be modeled by

$$C(g) = \begin{cases} 100 + 90\sqrt{g} & \text{if } 0 \le g \le 100 \\ 400 - 10e^5 + 6g + 10e^{0.05g} & \text{if } 100 < g \le 200 \end{cases}$$

and that for $0 \le g \le 200$, the revenue, in dollars, that she will bring in from selling g gallons of hot chocolate is given by

$$R(g) = 15g.$$

a. [4 points] For what quantities of hot chocolate sold would Gen's marginal revenue equal her marginal cost?

Solution: We have $R'(g) = 15$ and $C'(g) = \begin{cases} 45g^{-1/2} & \text{if } 0 \le g < 100 \\ 6 + 0.5e^{0.05g} & \text{if } 100 < g \le 200. \end{cases}$

For $0 \le g < 100$, marginal revenue is therefore equal to marginal cost when $45g^{-1/2} = 15$, so $g^{1/2} = 3$ and $g = 9$. When $100 < g \le 200$,

$$\begin{aligned} 6 + 0.5e^{0.05g} &= 15 \\ e^{0.05g} &= 18 \\ 0.05g &= \ln(18) \\ g &= 20\ln(18) \approx 57.8 \end{aligned}$$

However, this value of g is not in the domain of this piece, so MC and MR are never equal on this piece.

Note: We can also conclude that no such point exists on this interval by noting that since $6 + 0.5e^{0.05 \cdot 100} > 80$ and MC is increasing, MC never equal 15.

Answer: 9 **gallons**

b. [4 points] Assuming Gen can sell up to 200 gallons of hot chocolate, how much hot chocolate should she produce in order to maximize her profit, and what would that maximum profit be? *You must use calculus to find and justify your answer. Be sure to provide enough evidence to justify your answer fully.*

Solution: Note that $C(g)$ is continuous, since $100 + 90\sqrt{100} = 1000$ and $400 - 10e^5 + 6 \cdot 100 + 10e^{0.05 \cdot 100} = 1000$.

The profit function is given by $\pi(g) = R(g) - C(g)$. Since both $R(g)$ and $C(g)$ are continuous, we may, by the Extreme Value Theorem, consider only critical points and endpoints of the domain. The endpoints are at $g = 0$ and $g = 200$, and the critical points are at $g = 9$ (by previous part) and $g = 100$ (where MR is undefined).

$$\begin{aligned} \pi(0) &= 0 - 100 = -100 \\ \pi(9) &= 15 \cdot 9 - (100 + 90\sqrt{9}) = 135 - 370 = -235 \\ \pi(100) &= 1500 - 1000 = 500 \\ \pi(200) &\approx -217{,}380 \end{aligned}$$

(The last is because $\pi(200) = 15 \cdot 200 - (400 - 10e^5 + 6 \cdot 200 + 10e^{0.05 \cdot 200})$
$= 3000 - (400 - 10e^5 + 1200 + 10e^{10}) \approx 3000 - 220380 = -217{,}380$.)

Therefore the max occurs at $g = 100$, which results in a profit of \$500.

Answer: gallons of hot chocolate: 100 and max profit: \$500