5. [10 points] As a software engineer, Tendai spends many hours every day writing code. Let $w(t)$ be a function that models the number of lines of code that Tendai writes in a day if he works t hours that day. Tendai works at least one hour and at most 18 hours each day. A formula for $w(t)$ is given by

$$w(t) = \begin{cases} -2t^2 + 28t & \text{if } 1 \le t \le 3 \\ -0.5t^2 + 9t + 43.5 & \text{if } 3 < t \le 18. \end{cases}$$

a. [8 points] Find the values of t that minimize and maximize $w(t)$ on the interval $[1, 18]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE if appropriate.

Solution: Note that w is continuous at $t = 3$, since $\lim_{t\to 3^-} w(t) = \lim_{t\to 3^+} w(t) = 66$, so we may use the Extreme Value Theorem.
We find

$$w'(t) = \begin{cases} -4t + 28 & \text{if } 1 < t < 3 \\ -t + 9 & \text{if } 3 < t < 18. \end{cases}$$

The first expression is 0 when $t = 7$, but since this isn't in the domain of that piece, it is not a critical point. The second expression is 0 when $t = 9$.
Since both of these are polynomials, we don't have to worry about the derivative not existing on these open intervals. However, since $-4 \cdot 3 + 28 = 16$ and $-3 + 9 = 6$ are not equal, w' is not defined at 3, so $t = 3$ is also a critical point.
Computing $w(t)$ at each critical point and the endpoints gives:

t	1	3	9	18
$w(t)$	26	66	84	43.5

By the Extreme Value Theorem, we therefore find that $w(t)$ attains its maximum value at $t = 9$ and its minimum at $t = 1$.

Answer: global max(es) at $t =$ 9

Answer: global min(s) at $t =$ 1

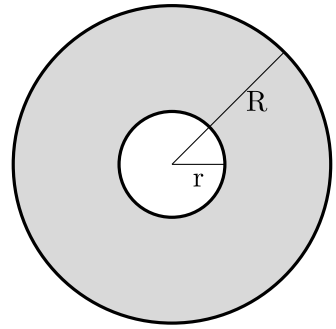
b. [2 points] What is the largest number of lines of code that Tendai can expect to write in a day according to this model?

Solution: From part **a.** we see that the maximum value of w is $w(9) = 84$. So according to this model, the largest number of lines of codes that Tendai can expect to write in a day is 84.

Answer: 84

8. [8 points]

Kristen is machining a metal washer to fix her broken down motorcycle. A washer is a flat, circular piece of metal with a hole in the middle. Kristen's washer is depicted by the shaded region in the figure to the right. The washer has an inner radius of r centimeters and an outer radius of R centimeters. The area of the washer must be exactly 5 square centimeters, and r must be at least 1 centimeter.



a. [3 points] Find a formula for r in terms of R.

Solution: The area of the washer is the difference between the outer circle's area and inner circle's area. So, since this must be 5 square centimeters we have $\pi R^2 - \pi r^2 = 5$, so $r^2 = \dfrac{\pi R^2 - 5}{\pi}$, and $r = \sqrt{\dfrac{\pi R^2 - 5}{\pi}}$.

Answer: $r =$ $\sqrt{\dfrac{\pi R^2 - 5}{\pi}}$

b. [2 points] The structural integrity of the washer depends on both its inner radius and its outer radius. Specifically, the structural integrity is given by the equation

$$S = 32R\left(\ln(rR + 1) + 7\right).$$

Express S as a function of R. *Your answer should not include r.*

Solution: We substitute our answer from part a. into the formula for S.

Answer: $S(R) =$ $32R(\ln\left(R\sqrt{\dfrac{\pi R^2 - 5}{\pi}} + 1\right) + 7)$

c. [3 points] What is the domain of $S(R)$ in the context of this problem? You may give your answer as an interval or using inequalities.

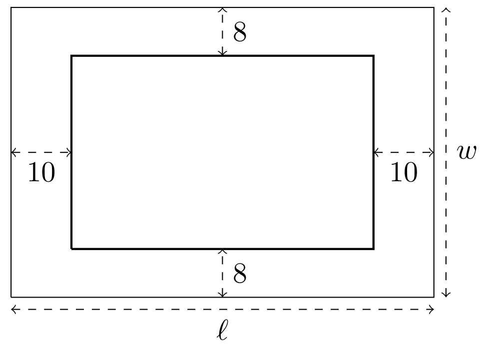
Solution: We are told that r must be at least 1. When $r = 1$, we have

$$\pi R^2 - \pi r^2 = 5$$
$$\pi R^2 - \pi = 5$$
$$R^2 = \frac{5 + \pi}{\pi}$$
$$R = \sqrt{\frac{5 + \pi}{\pi}}.$$

This is the smallest possible value of R, because if we make r larger, R must also be made larger so that the area of the washer can remain 5 square centimeters. There is no upper bound on how large R can be.

Answer: $[\sqrt{\dfrac{5 + \pi}{\pi}}, \infty)$

1. [7 points] Liam wants to build a rectangular swimming pool behind his new house. The pool will have an area of 1600 square feet. He will have 8-foot wide decks on two sides of the pool and 10-foot wide decks on the other two sides of the pool (see the diagram below).



a. [4 points] Let ℓ and w be the length and width (in feet) of the pool area including the decks as shown in the diagram. Write a formula for ℓ in terms of w.

Solution: The area of the pool needs to be 1600 sq ft, so

$$(\ell - 2(10))(w - 2(8)) = 1600$$

Solving this for ℓ gives

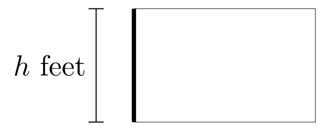
$$\ell = \frac{1600}{w-16} + 20$$

b. [3 points] Write a formula for the function $A(w)$ which gives the total area (in square feet) of the pool **and** the decks in terms of only the width w. Your formula should not include the variable ℓ. (This is the function Liam would minimize in order to find the minimum area that his pool and deck will take up in his yard. You do not need to do the optimization in this case.)

Solution: The pool and decks together make a rectangle of length ℓ and width w. The area A of the rectangle is $A = \ell w$. Substituting the formula from part (a) gives

$$A(w) = \left(\frac{1600}{w-16} + 20\right) w$$

1. [7 points] Gertrude wants to enclose a rectangular region in her backyard. She wants to use high fencing (thick line), which costs \$200 per foot, for one side of the rectangle. For the remaining three sides, she wants to use normal fencing (thin line), which costs \$75 per foot. Let $A(h)$ be the area (in square feet) of the region enclosed by the fence if h is the length (in feet) of the side with high fencing and Gertrude spends \$3000 on fencing for the project.



a. [4 points] Find a formula for $A(h)$.

Solution: Let ℓ be the other sidelength of the rectangle. Then, the total cost of the fencing is

$$200h + 75(2\ell + h) = 275h + 150\ell.$$

If the total cost of fencing is \$3000, then

$$\begin{aligned} 275h + 150\ell &= 3000 \\ 150\ell &= 3000 - 275h \\ \ell &= 20 - \frac{11}{6}h. \end{aligned}$$

Hence,

$$A(h) = h\ell = 20h - \frac{11}{6}h^2.$$

Answer: $A(h) =$ $20h - \dfrac{11}{6}h^2$

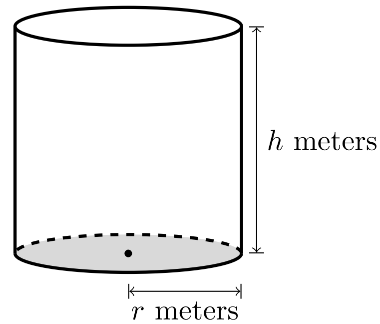
b. [3 points] In the context of this problem, what is the domain of $A(h)$?

Solution: Note that $h > 0$, or else we would not have a rectangle. Note also that $\ell > 0$ (where ℓ is the other sidelength).
So since $275h + 150\ell = 3000$, we have $275h = 3000 - 150\ell < 3000$, so $h < \frac{3000}{275} = \frac{120}{11} \approx 10.91$. Hence, the domain of $A(h)$ is $0 < h < \frac{120}{11}$.

(Note that in this situation, it would also be okay to include the endpoints 0 and 3000/275, which correspond to the degenerate cases of a rectangle of length or width 0.)

Answer: Domain: **The interval** $\left(0, \dfrac{120}{11}\right)$ **(or** $\left[0, \frac{120}{11}\right]$**)**

2. [10 points] Suma is making cylindrical paper cups that will be used to serve milkshakes at Qabil's Creamery. She rolls paper into a cylinder and then attaches it to the base. The thicker material that she uses for the base costs \$4.30 per square meter, and the lighter material that she uses for the vertical part of the cup costs \$2.20 per square meter. The radius of the circular base is r meters, and the height of the cup is h meters, as shown in the diagram on the right.
It may be helpful to know that the surface area of the vertical portion of the cup is $2\pi rh$.
Note: The top of the cup is left open.



Throughout this problem, assume that the material that Suma uses to make one paper cup costs \$0.12.

a. [4 points] Find a formula for h in terms of r.

Solution: The area of the vertical portion of the cup is $2\pi rh$ square meters, so the cost for the material for the vertical portion of one cup is $(2.20)(2\pi rh)$ dollars. Since the base of the cup is circular, its area is πr^2 square meters, and the cost for the material for the base of one cup is $(4.30)(\pi r^2)$ dollars. So the material that Suma uses to make one cup costs a total of $(2.20)(2\pi rh) + (4.30)(\pi r^2)$ dollars.
Therefore, we have $(2.20)(2\pi rh) + (4.30)(\pi r^2) = 0.12$.
Solving for h we find that $h = \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$.

Answer: $h = \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$

b. [2 points] Let $V(r)$ be the volume (in cubic meters) of the cup that Suma makes given that the material for the cup costs \$0.12 and the radius of the cup is r meters. Find a formula for $V(r)$. The variable h should not appear in your answer.
(Note: This is the function that Suma would use to find the value of r maximizing the volume of the cup, but you should not do the optimization in this case.)

Solution: Since Suma's cup is a cylinder, its volume is $\pi r^2 h$. So using what we found in part **a.** above, we see that $V(r) = \pi r^2 \cdot \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$ which simplifies to $V(r) = \dfrac{r(0.12 - 4.3\pi r^2)}{4.4}$.

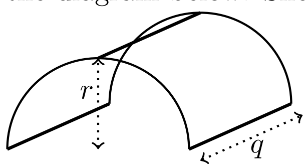
Answer: $V(r) = (\pi r^2) \cdot \dfrac{0.12 - 4.3\pi r^2}{4.4\pi r}$ **or** $\dfrac{r(0.12 - 4.3\pi r^2)}{4.4}$

c. [4 points] In the context of this problem, what is the domain of $V(r)$?

Solution: Since r is a length, it cannot be negative. Note also that if $r = 0$, then the cost of the materials for the cup would be 0 dollars (rather than \$0.12), so r must be positive. The height h also cannot be negative, and as h decreases, r increases. Therefore, r can certainly not be greater than when $h = 0$, in which case $4.3(\pi r^2) = 0.12$, so $r = \sqrt{\dfrac{0.12}{4.3\pi}}$ (since r must be positive). In this case, the cup would have height 0 and thus hold no milkshake, so we may choose to exclude this endpoint of the domain.

Answer: **the interval** $\left(0, \sqrt{\dfrac{0.12}{4.3\pi}}\right)$

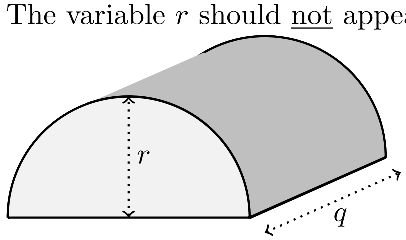
3. [9 points] Duncan's person is making him a new tent in the shape of half a cylinder. She plans to use wire to make the tent frame. This will consist of two semicircles of radius r (measured in inches) attached to three pieces of wire of length q (also measured in inches), as shown in the diagram below. She has 72 inches of wire to use for this.



a. [4 points] Find a formula for r in terms of q.

Solution: The amount of wire S used on the one semicircle of radius r is given by $S = \frac{1}{2}(2\pi r)$ inches. For the rest of the tent, she uses $3q$ inches of wire. Since she used 72 inches of wire to build the tent, we have $r = \dfrac{72 - 3q}{2\pi}$.

b. [2 points] Let $V(q)$ be the volume (in cubic inches) of the space inside the tent after the fabric is added, given that the total length of wire is 72 inches and the length of the tent is q inches. (Recall that the tent shape is half of a cylinder.) Find a formula for $V(q)$. The variable r should <u>not</u> appear in your answer.



Solution: The volume of enclosed by the tent V is the volume of a half cylinder. In this case $V = \frac{1}{2}\pi r^2 h$, where r is the radius of the semicircular lateral face and h is the length of the tent. In our case $h = q$ and using our answer from part **a** we obtain

$$V = \frac{1}{2}\pi r^2 h = \frac{\pi q}{2}\left(\frac{72 - 3q}{2\pi}\right)^2$$

c. [3 points] In the context of this problem, what is the domain of $V(q)$?

Solution: The length q of the tent cannot be negative and it has to be smaller than the total amount of wire used 72 inches, then $0 < q < 72$. On the other hand, the more wire you use building the length of the tent q the smallest the radius r will be. If we set our expression for r in terms of q from **a.** to 0: $\dfrac{72 - 3q}{2\pi} = 0$, we obtain $3q = 72$ and therefore $q = 24$. Hence the domain of $V(q)$ is all values of q that satisfy $0 < q < 24$.

4. [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that t seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where k and c are nonzero constants.

a. [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of t. The constants k and c may appear in your answer.

Solution: The velocity is the derivative of the height function, so we compute

$$v(t) = h'(t) = -15k\sin(kt).$$

Notice that the Chain Rule gives us a factor of k out front, and since c is an additive constant, it disappears when we take the derivative.
Notice also that $v(t) = \frac{dh}{dt}$ does indeed have units of feet per second, as required.

Answer: $v(t) =$ $-15k\sin(kt)$

b. [2 points] Find a formula for $v'(t)$. The constants k and c may appear in your answer.

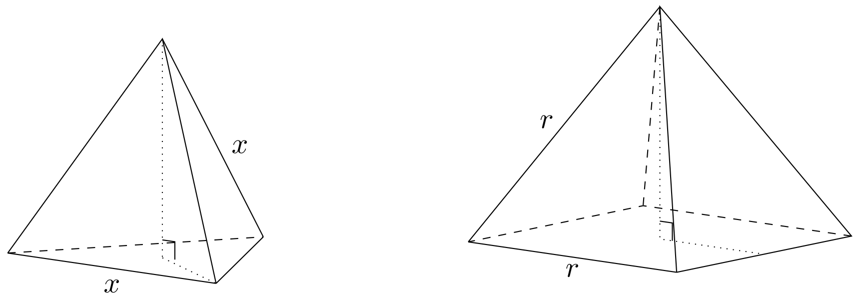
Answer: $v'(t) =$ $-15k^2\cos(kt)$

c. [3 points] What is the maximum vertical acceleration experienced by the captain? The constants k and c may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

Solution: The acceleration is just the derivative of the velocity function, which was just computed in the previous part.
Since $v'(t) = -15k^2\cos(kt)$ is sinusoidal with midline 0 and amplitude $15k^2$, the maximum value it achieves is $15k^2$.
Since $v'(t) = \frac{dv}{dt}$, the units on the acceleration are feet per second per second, or feet per second squared.

Answer: Max vertical acceleration: $15k^2$ **ft/s^2**

5. [7 points] An alien is building the wire frames of two pyramids. One has a base that is an equilateral triangle with side length x meters, and the other has a base that is a square with side length r meters. These shapes are shown below. For both, all triangular faces are equilateral.



Triangular Pyramid Square Pyramid

The alien has 2 meters of wire available to build the frames, and **will use all of it**.

a. [2 points] Find a formula for r in terms of x.

Solution: There are 6 sides of length x meters, and 8 sides of length r meters. In total, these lengths must add up to 2 meters, so $6x + 8r = 2$. We can then solve for r in terms of x and find that $r = \frac{2-6x}{8}$.

Answer: $r = \dfrac{2-6x}{8}$

b. [3 points] Find a formula for $A(x)$, the combined surface area of the two pyramids (i.e. the total area of all sides and bases of both shapes). Your formula should be in terms of x only.

Recall that the area of an equilateral triangle with side length L is $\dfrac{\sqrt{3}}{4}L^2$.

Solution: On the triangular pyramid, there are 4 equilateral triangles each with side length x. On the square pyramid, there are 4 equilateral triangles each with side length r, plus one square of side length r on the base. Adding up the areas of these shapes, we find

$$\text{Total Surface Area} = 4\left(\frac{\sqrt{3}}{4}x^2\right) + 4\left(\frac{\sqrt{3}}{4}r^2\right) + r^2 = \sqrt{3}x^2 + \sqrt{3}r^2 + r^2 = \sqrt{3}x^2 + (\sqrt{3}+1)r^2.$$

We substitute $r = \dfrac{2-6x}{8}$ and find $A(x) = \sqrt{3}x^2 + \left(\sqrt{3}+1\right)\left(\frac{2-6x}{8}\right)^2$

Answer: $A(x) = \sqrt{3}x^2 + \left(\sqrt{3}+1\right)\left(\dfrac{2-6x}{8}\right)^2$

c. [2 points] The alien wants to actually build one of each type of pyramid. In the context of the problem, what is the domain of the function $A(x)$ from part **b.**? You may give your answer as an interval or using inequalities.

Solution: We must have $x > 0$ in order to get a triangular pyramid. We also need $r > 0$ to get a square pyramid, which in terms of x means

$$\frac{2-6x}{8} > 0 \quad \text{which simplifies to} \quad 2 > 6x, \quad \text{so} \quad \frac{1}{3} > x$$

Answer: $\left(0, \dfrac{1}{3}\right)$