Optimizations and Modeling

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1 General strategy

Step 1 Identify what quantity to be optimized

Step 2 Find a formula for the function of the quantity. If necessary, eliminate from this formula all but one variable. Identify the domain over which this variable varies.

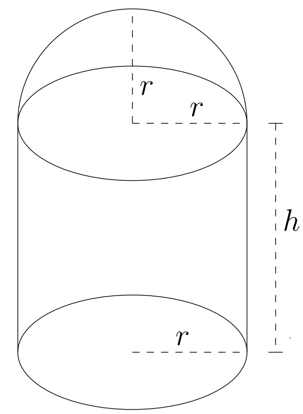
Step 3 Optimization – Find critical points and evaluate the function at these points and the endpoints (if relevant) to find the global maxima and/or minima.

2 Geometrical Facts

http://www.math.lsa.umich.edu/courses/115/Exams/Exam_2/Materials/Math115GeometryFormulasW20.pdf

3. Daisy wants to build a new silo on her farm. The sides of the silo will be a cylinder of height h feet and radius r feet, and its roof will be a hemisphere of radius r feet, as shown in the picture below. (She does not need to construct a floor.) Suppose that the material for constructing the sides of the cylinder costs $\frac{6}{\pi}$ dollars per square foot while the material needed to construct the roof costs $\frac{20}{\pi}$ dollars per square foot.

   Recall that the sides of a cylinder, without the circular ends, have area $2\pi rh$. Also note that a sphere has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.



   (a) If Daisy wants to spend 162,000 dollars constructing her silo, find a formula for h in terms of r.

   (b) Find a formula for the volume of the silo V, in cubic feet, in terms only of the variable r. Your answer should not include the variable h.

   (c) Find the domain of the function $V(r)$.

   (d) Find the values of r and h which maximize the volume of the silo.

   (e) Suppose now that we add the following constraints:

   - To ensure that there is sufficient space between the silo and nearby buildings, the diameter of the silo can be at most 110 feet.
   - A county ordinance forbids the construction of buildings more than 225 feet high.

   How does this change the domain of $V(r)$? Find the values of r and h which maximize the volume of the silo under these constraints.

   Hint: Note that the total height of the silo is <u>not</u> h.

5. [10 points] As a software engineer, Tendai spends many hours every day writing code. Let $w(t)$ be a function that models the number of lines of code that Tendai writes in a day if he works t hours that day. Tendai works at least one hour and at most 18 hours each day. A formula for $w(t)$ is given by

$$w(t) = \begin{cases} -2t^2 + 28t & \text{if } 1 \le t \le 3 \\ -0.5t^2 + 9t + 43.5 & \text{if } 3 < t \le 18. \end{cases}$$

a. [8 points] Find the values of t that minimize and maximize $w(t)$ on the interval $[1, 18]$. Use calculus to find your answers, and be sure to show enough evidence that the points you find are indeed global extrema. For each answer blank, write NONE if appropriate.

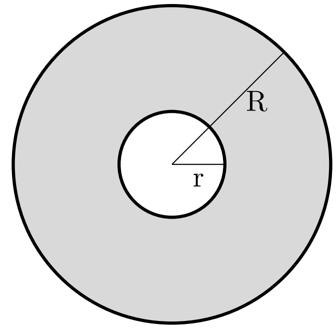
Answer: global max(es) at $t =$ ____________________

Answer: global min(s) at $t =$ ____________________

b. [2 points] What is the largest number of lines of code that Tendai can expect to write in a day according to this model?

Answer: ____________________

8. [8 points]



Kristen is machining a metal washer to fix her broken down motorcycle. A washer is a flat, circular piece of metal with a hole in the middle. Kristen's washer is depicted by the shaded region in the figure to the right. The washer has an inner radius of r centimeters and an outer radius of R centimeters. The area of the washer must be exactly 5 square centimeters, and r must be at least 1 centimeter.

a. [3 points] Find a formula for r in terms of R.

Answer: $r =$ ________________________

b. [2 points] The structural integrity of the washer depends on both its inner radius and its outer radius. Specifically, the structural integrity is given by the equation

$$S = 32R\left(\ln(rR+1)+7\right).$$

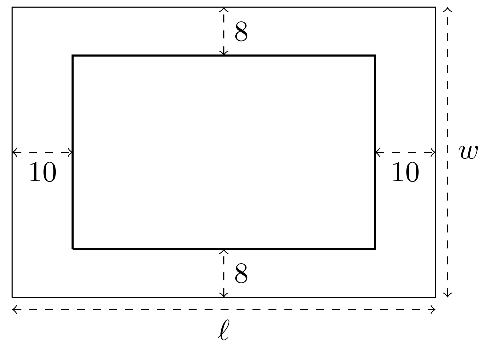
Express S as a function of R. *Your answer should not include r.*

Answer: $S(R) =$ ________________________

c. [3 points] What is the domain of $S(R)$ in the context of this problem? You may give your answer as an interval or using inequalities.

Answer: ________________________

1. [7 points] Liam wants to build a rectangular swimming pool behind his new house. The pool will have an area of 1600 square feet. He will have 8-foot wide decks on two sides of the pool and 10-foot wide decks on the other two sides of the pool (see the diagram below).



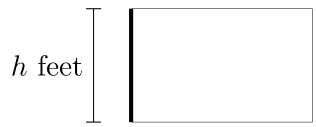
a. [4 points] Let ℓ and w be the length and width (in feet) of the pool area including the decks as shown in the diagram. Write a formula for ℓ in terms of w.

$\ell =$ ______________________

b. [3 points] Write a formula for the function $A(w)$ which gives the total area (in square feet) of the pool **and** the decks in terms of only the width w. Your formula should not include the variable ℓ. (This is the function Liam would minimize in order to find the minimum area that his pool and deck will take up in his yard. You do not need to do the optimization in this case.)

$A(w) =$ ______________________

1. [7 points] Gertrude wants to enclose a rectangular region in her backyard. She wants to use high fencing (thick line), which costs \$200 per foot, for one side of the rectangle. For the remaining three sides, she wants to use normal fencing (thin line), which costs \$75 per foot. Let $A(h)$ be the area (in square feet) of the region enclosed by the fence if h is the length (in feet) of the side with high fencing and Gertrude spends \$3000 on fencing for the project.



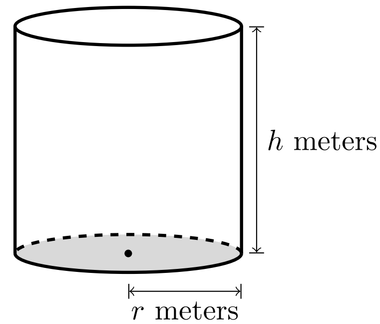
a. [4 points] Find a formula for $A(h)$.

Answer: $A(h) =$ ____________________________

b. [3 points] In the context of this problem, what is the domain of $A(h)$?

Answer: Domain: ____________________________

2. [10 points] Suma is making cylindrical paper cups that will be used to serve milkshakes at Qabil's Creamery. She rolls paper into a cylinder and then attaches it to the base. The thicker material that she uses for the base costs \$4.30 per square meter, and the lighter material that she uses for the vertical part of the cup costs \$2.20 per square meter. The radius of the circular base is r meters, and the height of the cup is h meters, as shown in the diagram on the right.
It may be helpful to know that the surface area of the vertical portion of the cup is $2\pi rh$.
Note: The top of the cup is left open.



Throughout this problem, assume that the material that Suma uses to make one paper cup costs \$0.12.

a. [4 points] Find a formula for h in terms of r.

Answer: $h =$ ______________________________

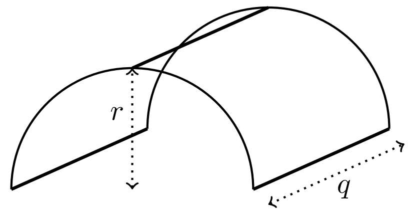
b. [2 points] Let $V(r)$ be the volume (in cubic meters) of the cup that Suma makes given that the material for the cup costs \$0.12 and the radius of the cup is r meters. Find a formula for $V(r)$. The variable h should <u>not</u> appear in your answer.
(Note: This is the function that Suma would use to find the value of r maximizing the volume of the cup, but you should <u>not</u> do the optimization in this case.)

Answer: $V(r) =$ ______________________________

c. [4 points] In the context of this problem, what is the domain of $V(r)$?

Answer: ______________________________

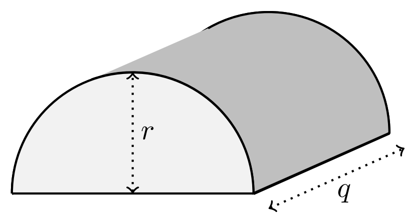
3. [9 points] Duncan's person is making him a new tent in the shape of half a cylinder. She plans to use wire to make the tent frame. This will consist of two semicircles of radius r (measured in inches) attached to three pieces of wire of length q (also measured in inches), as shown in the diagram below. She has 72 inches of wire to use for this.



a. [4 points] Find a formula for r in terms of q.

Answer: $r =$ ______________________

b. [2 points] Let $V(q)$ be the volume (in cubic inches) of the space inside the tent after the fabric is added, given that the total length of wire is 72 inches and the length of the tent is q inches. (Recall that the tent shape is half of a cylinder.) Find a formula for $V(q)$. The variable r should <u>not</u> appear in your answer.
(Note: This is the function that Duncan's person would use to find the value of q that maximizes the volume of the tent, but you should not do the optimization in this case.)



Answer: $V(q) =$ ______________________

c. [3 points] In the context of this problem, what is the domain of $V(q)$?

Answer: ______________________

4. [8 points] A ship's captain is standing on the deck while sailing through stormy seas. The rough waters toss the ship about, causing it to rise and fall in a sinusoidal pattern. Suppose that t seconds into the storm, the height of the captain, in feet above sea level, is given by the function

$$h(t) = 15\cos(kt) + c$$

where k and c are nonzero constants.

a. [3 points] Find a formula for $v(t)$, the vertical velocity of the captain, in feet per second, as a function of t. The constants k and c may appear in your answer.

Answer: $v(t) =$ ____________________________

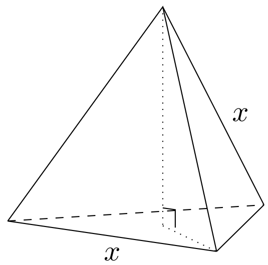
b. [2 points] Find a formula for $v'(t)$. The constants k and c may appear in your answer.

Answer: $v'(t) =$ ____________________________

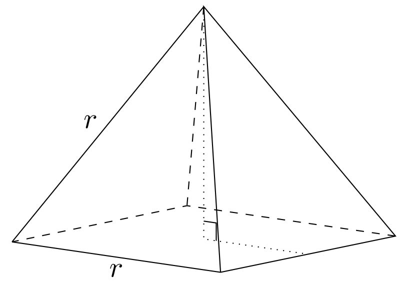
c. [3 points] What is the maximum vertical acceleration experienced by the captain? The constants k and c may appear in your answer. You do not need to justify your answer or show work. *Remember to include units.*

Answer: Max vertical acceleration: ________________________

5. [7 points] An alien is building the wire frames of two pyramids. One has a base that is an equilateral triangle with side length x meters, and the other has a base that is a square with side length r meters. These shapes are shown below. For both, all triangular faces are equilateral.



Triangular Pyramid



Square Pyramid

The alien has 2 meters of wire available to build the frames, and **<u>will use all of it</u>**.

a. [2 points] Find a formula for r in terms of x.

Answer: $r =$ ________________________

b. [3 points] Find a formula for $A(x)$, the combined surface area of the two pyramids (i.e. the total area of all sides and bases of both shapes). Your formula should be in terms of x only. Recall that the area of an equilateral triangle with side length L is $\frac{\sqrt{3}}{4}L^2$.

Answer: $A(x) =$ ________________________

c. [2 points] The alien wants to actually build one of each type of pyramid. In the context of the problem, what is the domain of the function $A(x)$ from part **b.**? You may give your answer as an interval or using inequalities.

Answer: ________________________