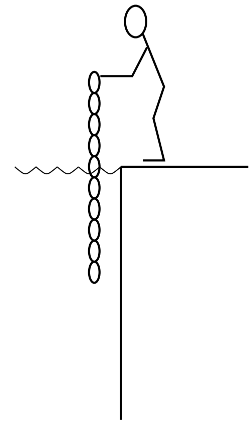
8. [16 points] Chris is standing at the edge of a swimming pool, holding a chain that is partially submerged in the water of the pool, as shown in the figure to the right. The chain is six feet long and weighs 5 lb/ft. When it is in the water, however, the buoyant force of the water makes the effective weight of the chain less—in the water, it weighs only 3 lb/ft. If the chain is initially half submerged in the pool and Chris lifts it straight up until it is entirely out of the water, how much work does Chris do?



Solution:
We can find the total work by considering the work to move from a given position, x (measured as the distance that the chain has been raised), to the position $x + \Delta x$. The required force is the weight of the chain,

$$F = (\text{weight of chain above the water}) + (\text{weight of chain in the water}).$$

The length of chain above the water is $3 + x$ ft, and the length below is $3 - x$ ft. Thus $F = (3 + x)(5) + (3 - x)(3) = 24 + 2x$ lb. The work to lift the chain through this distance Δx is then $\Delta W = (24 + 2x)\Delta x$. The total work is found by integrating over the 3 ft that it is lifted, so

$$W = \int_0^3 (24 + 2x)\,dx = (24x + x^2)\Big|_0^3 = 72 + 9 = 81\,\text{ft}\cdot\text{lb}.$$

An alternate solution is to consider the top half and bottom half of the chain separately. The top half all moves 3 ft and has a constant weight, so the work is $W_t = ((3\,\text{ft})(5\,\text{lb/ft}))(3\,\text{ft}) = 45\,\text{ft}\cdot\text{lb}$. Then, a piece of the bottom part of chain that has length Δx and is x feet from the bottom of the chain's initial position has a weight 3 lb/ft for the distance $3 - x$ and a weight 5 lb/ft for the distance x. Thus the work to lift it is $\Delta W = (3\Delta x)(3 - x) + (5\Delta x)x = (9 + 2x)\Delta x$. The total work to lift the bottom half of the chain is then $W_b = \int_0^3 9 + 2x\,dx = 9x + x^2\big|_0^3 = 27 + 9 = 36\,\text{ft}\cdot\text{lb}$. The total work is the sum of W_t and W_b, which is not surprisingly still 81 ft·lb.