Math 116 — Practice for Exam 3

Generated December 6, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: ________

1. This exam has 7 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2016	2	8	hiking	13	
Winter 2015	3	9	ladybugs1	10	
Winter 2015	2	7	black hole	9	
Fall 2014	2	1	robot uprising	12	
Fall 2011	3	9	bank	11	
Winter 2010	2	9	ice cube	14	
Fall 2009	3	4	ice sculpture	12	
Total				81	

Recommended time (based on points): 83 minutes

8. [13 points] Brianne is hiking, and the temperature of the air in °C after she's traveled x km is a solution to the differential equation

$$y' + y \sin x = 0$$

a. [7 points] Find the general solution of the differential equation.

Solution: Writing the equation as $\frac{dy}{dx} = -y \sin x$ and separating the variables we get

$$\int \frac{1}{y} dy = \int -\sin x dx$$

$$\ln |y| = \cos x + C$$

$$y = Ae^{\cos x}$$

b. [2 points] If the temperature was 10 °C at the beginning of the hike, find $T(x)$, the temperature of the air in °C after she's traveled x km. Show your work.

Solution: From (a), $T(x) = Ae^{\cos x}$. Since $T(0) = 10$ we get $A = \frac{10}{e}$. Thus,

$$T(x) = \frac{10}{e} e^{\cos x}$$

c. [4 points] Brianne traveled 7 km on the hike. Using the information given in (b), find the coldest air temperature she encountered on the hike. Give an **exact** answer (i.e. no decimal approximations).

Solution: We want to find the minimum of the function $T(x) = \frac{10}{e} e^{\cos x}$ over the interval $[0, 7]$. The critical points are $0, \pi, 2\pi$. Checking the outputs of T at those points and the endpoint 7 we find that the minimum is $T(\pi) = \frac{10}{e^2}$.

9. [10 points] Vic is planning to put ladybugs in his garden to eat harmful pests. The ladybug expert at the gardening store claims that the number of ladybugs in his garden can be modeled by the differential equation

$$\frac{dL}{dt} = \frac{L}{20} - \frac{L^2}{100}$$

where L is the number of ladybugs, in hundreds, in Vic's garden, t days after they are introduced.

a. [4 points] Find the equilibrium solutions to this differential equation and indicate their stability.

Solution: To find the equilibrium solutions we set the right hand side of the equation above equal to 0. The resulting equation simplifies to $L(5 - L) = 0$. There is then an unstable equilibrium at $L = 0$ and a stable equilibrium at $L = 5$.

b. [2 points] If Vic starts his garden with 50 ladybugs, what will the long term population of ladybugs in his garden be according to the differential equation above?

Solution: The long term population is 500 ladybugs, as the solution to the differential equation above with initial condition $L(0) = .5$ will tend to the stable equilbrium at $L = 5$ as $t \to \infty$.

The long term population is **500 ladybugs**

c. [4 points] For what value of b is the function $L(t) = 5e^{bt}\left(4 + e^{bt}\right)^{-1}$ a solution to this differential equation.

Solution: We can compute that $\dfrac{dL}{dt} = 5be^{bt}\left(4 + e^{bt}\right)^{-1} - 5e^{2bt}\left(4 + e^{bt}\right)^{-2} = \dfrac{20be^{bt}}{(1 + e^{bt})^2}$ and $\dfrac{L}{20} - \dfrac{L^2}{100} = \dfrac{5e^{bt}\left(4 + e^{bt}\right)^{-1}}{20} - \dfrac{25e^{2bt}\left(4 + e^{bt}\right)^{-2}}{100} = \dfrac{e^{bt}}{(1 + e^{bt})^2}$. These two expressions will be equal provided that $b = \dfrac{1}{20}$.

$b =$ **1/20**

7. [9 points] A certain cosmological model predicts the evaporation rate of a black hole to be inversely proportional to its mass squared. This gives a first order differential equation

$$\frac{dM}{dt} = \alpha \frac{1}{M^2}$$

where $M = M(t)$ is the mass of the black hole in kg, t is time in seconds, and α is the constant of proportionality.

a. [5 points] Find the general solution using separation of variables.

Solution: We multiply both sides of the equation by M^2 and then integrate both sides with respect to t,

$$\int M^2 \frac{dM}{dt} dt = \int \alpha dt$$

$$\frac{M^3}{3} = \alpha t + C$$

$$M = (3\alpha t + C)^{1/3}$$

where C is an arbitrary constant.

b. [4 points] How long will it take for a black hole with initial mass 8×10^{22} kg, which is approximately the mass of the moon, to evaporate if $\alpha = -\frac{8}{3} \times 10^{17}$ kg^3/sec?

Solution: If $M(0) = 8 \times 10^2 2$ kg, then $M = \left(3\alpha t + (8 \times 10^{22})^3\right)^{1/3}$.To find the evaporation time, we set $M = 0$ and solve for t. Setting $M = 0$ gives us the equation

$$3\alpha t + \left(8 \times 10^{22}\right)^3 = 0.$$

Solving for t, we have

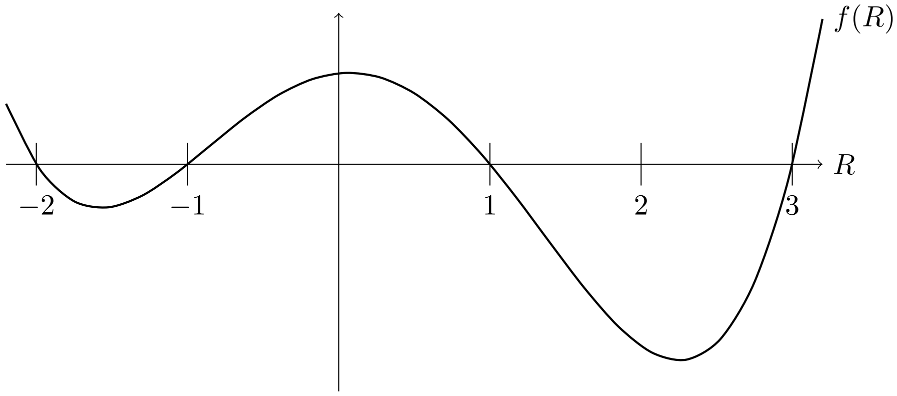
$$t = \frac{\left(8 \times 10^{22}\right)^3}{-3\alpha} = 64 \times 10^{49}.$$

Thus, it will take 64×10^{49} seconds for the black hole to evaporate.

1. [12 points] Franklin, your robot, is on the local news. Let $R(t)$ be the number of robots, in millions, that have joined the robot uprising t minutes after the start of the broadcast. After watching the news for a little bit, you find that $R(t)$ obeys the differential equation:

$$\frac{dR}{dt} = f(R)$$

for some function $f(R)$. A graph of $f(R)$ is shown below.



a. [3 points] If $R(t)$ is the solution to the above differential equation with $R(0) = 0$, what is $\lim_{t\to\infty} R(t)$? Justify your answer.

Solution: If $R = 0$, $f(R) = R'(t)$ is positive, so R will increase as t increases. As R increases to 1, $R'(t) = f(R)$ goes to 0, so $\lim_{t\to\infty} R(t) = 1$.

b. [6 points] Find the equilibrium solutions to the above differential equation **and** classify them as stable or unstable.

Solution:

$R = -2$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = -1$	**Stable**	$\boxed{\textbf{Unstable}}$
$R = 1$	$\boxed{\textbf{Stable}}$	**Unstable**
$R = 3$	**Stable**	$\boxed{\textbf{Unstable}}$

c. [3 points] Let $R(t)$ be a solution to the given differential equation, with $R(3) = 0.5$. Is the graph of $R(t)$ concave up, concave down, or neither at the point $(3, 0.5)$? Justify your answer.

Solution:

$$\frac{d^2R}{dt^2} = \frac{d}{dt}f(R) = f'(R)\frac{dR}{dt} = f'(R)f(R)$$

At $R = 0.5$, $f'(0.5) < 0$ and $f(0.5) > 0$ so $\frac{d^2R}{dt^2} < 0$. Therefore, the solution curve will be concave down.

9. [11 points] Philip J. Frye has a bank account with Big Apple Bank that compounds interest at a continuous annual rate of 1%. His account has a balance of \$300 at midnight of January 1, 2000, when Frye is cryogenically frozen for 1000 years. The entire time he is frozen, his account accumulates interest. Include units in your answers where appropriate.

a. [2 points] Write a differential equation that models Frye's account balance $M(t)$, where M is given in dollars and t is the number of years since January 1, 2000. List any initial conditions.

Solution:

$$\frac{dM}{dt} = 0.01M \qquad M(0) = 300.$$

b. [4 points] Solve your differential equation from (a) to find the balance in Frye's account after he is awoken in the year 3000. Show all your work.

Solution:

$$\frac{dM}{dt} = 0.01M \qquad \frac{dM}{M} = 0.01dt \qquad \ln|M| = 0.01t + C$$

$$M(t) = Be^{0.01t}$$

$$M(0) = 300 \qquad 300 = B \qquad M(t) = 300e^{0.01t}.$$

c. [2 points] Suppose that Big Apple Bank charges an annual fee of \$5 to maintain the account, withdrawn continuously over the course of the year. Write a new differential equation for $M(t)$, the balance in Frye's bank account.

Solution:

$$\frac{dM}{dt} = 0.01M - 5.$$

d. [3 points] How large must the initial deposit in Frye's account be at Big Apple Bank in order for the account to be profitable for him? Justify your answer mathematically.

Solution: The differential equation has an equilibrium solution at $M_{eq} = \frac{5}{0.01} = 500$. The equilibrium solution is unstable since

$$\left.\frac{dM}{dt}\right|_{M=400} = .01(400) - 5 = -1 < 0 \qquad \text{and} \quad \left.\frac{dM}{dt}\right|_{M=600} = .01(600) - 5 = 1 > 0$$

Hence the initial deposit $M(0)$ has to be larger than $M_{eq} = 500$ dollars in order to be profitable ($M(0) > 500$).

9. [14 points] An ice cube melts at a rate proportional to its surface area. Let $V(t)$ denote the volume (in cm^3) of the ice cube, and let $x(t)$ denote the length (in cm) of a side of the ice cube t seconds after it begins to melt.

a. [4 points] Write a differential equation for $V(t)$, the ice cube's volume t seconds after it started melting. Your differential equation may contain V, t and an unknown constant k.

Solution: We know that $V = x^3$, so $x = V^{1/3}$. The surface area of the cube is given by $6x^2$. That gives us $\frac{dV}{dt} = 6kx^2$, and substituting x in terms of V, we have $\frac{dV}{dt} = 6kV^{2/3}$.

b. [4 points] The ice cube's initial volume is $V_0 > 0$. Solve the differential equation you found in part (a), finding V in terms of t, k, and V_0.

Solution: Using separation of variables, we have $\frac{dV}{V^{2/3}} = 6kdt$. This gives $3V^{1/3} = 6kt + C$, and $V^{1/3} = 2kt + C$, or $V = (2kt + C)^3$. When $t = 0$, $V = V_0$, which gives us $V_0 = C^3$, so that $C = V_0^{1/3}$. The solution is then $V = (2kt + V_0^{1/3})^3$.

c. [6 points] Graph the volume of the ice cube versus time given $V(0) = V_0$. Be sure to label your axes and any important features of your graph, including the time at which the ice cube has completely melted.

Solution: The vertical intercept is $V = V_0$. The horizontal intercept is $t = -\frac{1}{2k}V_0^{1/3}$.

4. [12 points] During a party, the host discovers that he has been robbed of his favorite gold hallway clock, and immediately calls the police. The last time the host noticed the clock was still hanging, it was 6:00 p.m. When the police arrive, Officer Tom notices a decorative ice sculpture. When he first arrives at 9:00 p.m., the ice sculpture's temperature is 27°F. After questioning others at the party, Officer Tom again takes the temperature of the ice sculpture at 10:00 p.m., and finds it to be 29°F. The temperature of the room has remained a constant 68°F all day.

a. [2 points] Let S be the temperature of the sculpture, measured in degrees Fahrenheit. Assuming the sculpture obeys Newton's Law of Heating and Cooling, write a differential for $\frac{dS}{dt}$, where t is the number of hours since 9:00 p.m. Your answer may contain an unknown constant, k.

Solution: $\frac{dS}{dt} = k(S - 68)$ (or some appropriate variation thereof)

b. [7 points] Using separation of variables, and the information provided about the sculpture, solve the differential equation to find $S(t)$, where t is the number of hours since 9:00 p.m. Your answer should contain no unknown constants.

Solution: $\frac{dS}{S-68} = kdt$, which gives $\ln|S - 68| = kt + C$, or $S = Ae^{kt} + 68$. When $t = 0$, $S = 27$, which gives us $27 = A + 68$, so $A = -41$, leaving $S = -41e^{kt} + 68$. We use the other condition to solve for k. At $t = 1$, $S = 29$, so $29 = -41e^{k(1)} + 68$. We solve for k: $41e^k = 39$, or $e^k = \frac{39}{41}$, which leaves us with $k = \ln\left(\frac{39}{41}\right) \approx -0.05001$.

$$S = -41e^{-0.05001t} + 68$$

c. [3 points] Company FunIce provided the sculpture, and it was delivered by their employee Bill. For each ice sculpture they produce, the company guarantees the temperature will be exactly 18°F upon delivery. If the sculpture was 18°F at delivery, and Bill left directly afterwards, should he be considered as a possible suspect of the robbery? Briefly justify your answer.

Solution: We solve for when the temperature of the sculpture was $S = 18$. We solve for t: $18 = -41e^{-0.05001t} + 68$, or $\frac{50}{41} = e^{-0.05001t}$. This gives us $t \approx -3.96823$ hours. The sculpture was delivered nearly four hours before 9:00 p.m., putting the delivery near 5:00 p.m. Since Bill left immediately after delivery, he should not be considered as a suspect, since the clock was still there at 6:00 p.m.