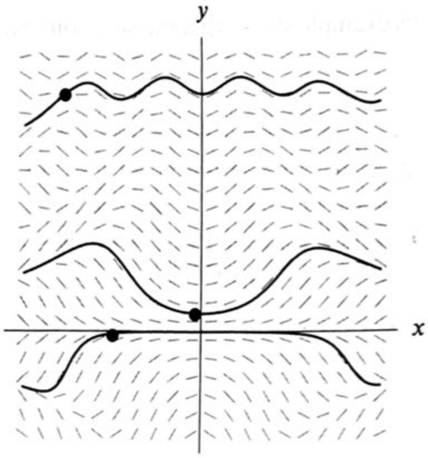


Figure 11.13: There's one and only one solution curve through each point in the plane for this slope field (dots represent initial conditions).

In the language of differential equations, an initial value problem (that is, a differential equation and an initial condition) almost always has a unique solution. One way to see this is by looking at the slope field. Imagine starting at the point representing the initial condition. Through that point there is usually a line segment pointing in the direction of the solution curve. By following these line segments, we trace out the solution curve. See Figure 11.13. In general, at each point there is one line segment and therefore only one direction for the solution curve to go. The solution curve *exists* and is *unique* provided we are given an initial point. Notice that even though we can draw the solution curves, we may have no simple formula for them.

It can be shown that if the slope field is continuous as we move from point to point in the plane, we can be sure that a solution curve exists everywhere. Ensuring that each point has only one solution curve through it requires a slightly stronger condition.

Exercises and Problems for Section 11.2 Online Resource: Additional Problems for Section 11.2

EXERCISES

1. **(a)** For $dy/dx = x^2 - y^2$, find the slope at the following points:

$$(1,0), \quad (0,1), \quad (1,1), \quad (2,1), \quad (1,2), \quad (2,2)$$

(b) Sketch the slope field at these points.

2. Sketch the slope field for $dy/dx = x/y$ at the points marked in Figure 11.14.

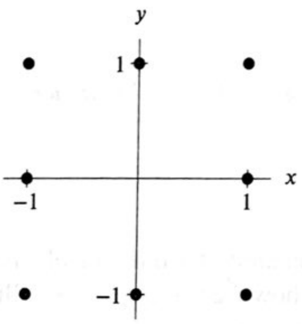


Figure 11.14

3. Sketch the slope field for $dy/dx = y^2$ at the points marked in Figure 11.15.

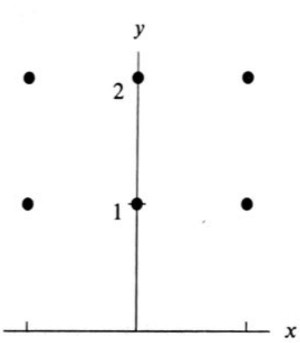
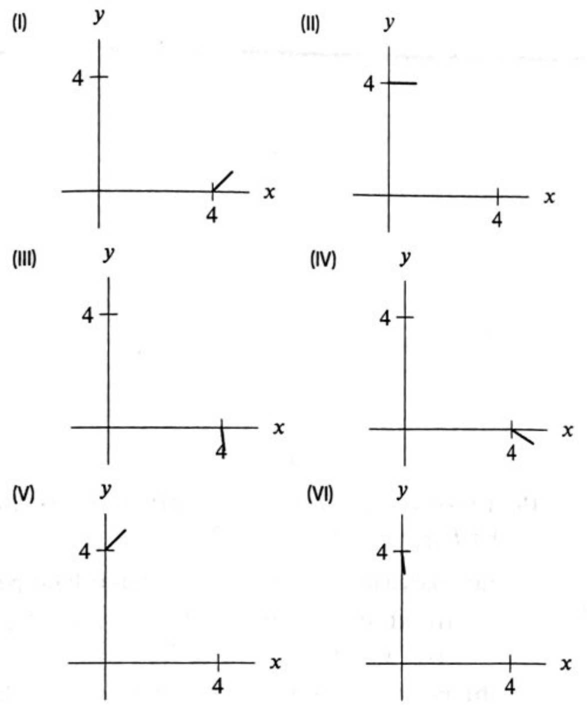


Figure 11.15

4. Match each of the slope field segments in (I)–(VI) with one or more of the differential equations in (a)–(f).

(a) $y' = e^{-x^2}$ **(b)** $y' = \cos y$
(c) $y' = \cos(4 - y)$ **(d)** $y' = y(4 - y)$
(e) $y' = y(3 - y)$ **(f)** $y' = x(3 - x)$



5. **Sketch three solution curves for each of the slope fields in Figures 11.16 and 11.17.**

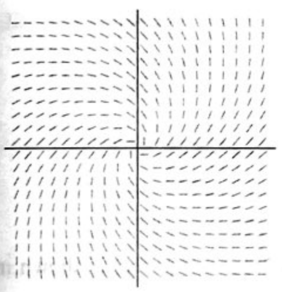


Figure 11.16

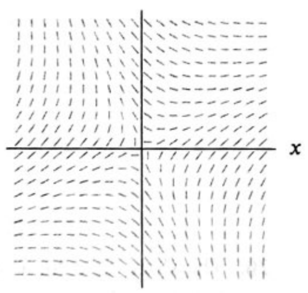


Figure 11.17

6. Sketch three solution curves for each of the slope fields in Figure 11.18.

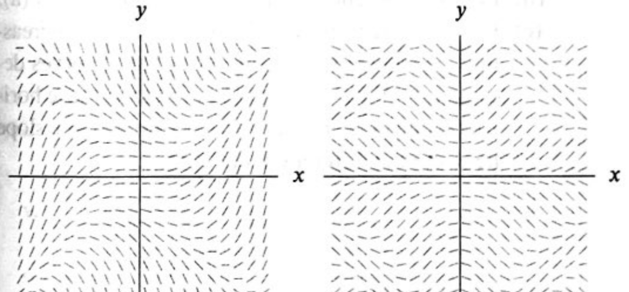


Figure 11.18

■ For Exercises 7–9, at the given point, is the solution curve to $y' = 2y - 3x - 4$ increasing or decreasing?

7. $(1, 4)$ 8. $(2, 4)$ 9. $(0, 3)$

10. One of the slope fields in Figure 11.18 is the slope field for $y' = x^2 - y^2$. Which one? On this field, where is the point $(0, 1)$? The point $(1, 0)$? (Assume that the x and y scales are the same.) Sketch the line $x = 1$ and the solution curve passing through $(0, 1)$ until it crosses $x = 1$.

11. The slope field for the equation $y' = x(y - 1)$ is shown in Figure 11.19.

(a) Sketch the solutions that pass through the points

(i) $(0, 1)$ (ii) $(0, -1)$ (iii) $(0, 0)$

(b) From your sketch, write the equation of the solution with $y(0) = 1$.

(c) Check your solution to part (b) by substituting it into the differential equation.

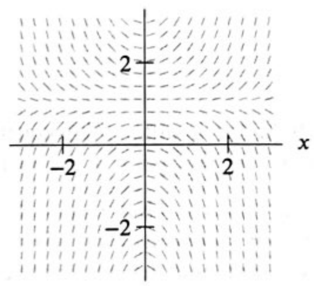


Figure 11.19

12. The slope field for the equation $y' = x + y$ is shown in Figure 11.20.

(a) Sketch the solutions that pass through the points

(i) $(0, 0)$ (ii) $(-3, 1)$ (iii) $(-1, 0)$

(b) From your sketch, write the equation of the solution passing through $(-1, 0)$.

(c) Check your solution to part (b) by substituting it into the differential equation.

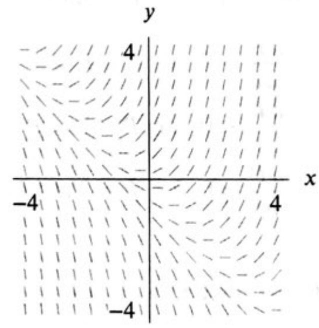


Figure 11.20: Slope field for $y' = x + y$

PROBLEMS

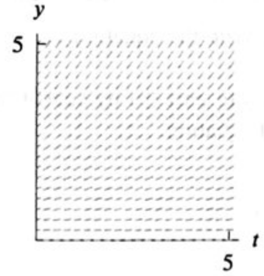
13. **(a)** Match the slope fields (A) and (B) with the following differential equations:

(i) $y' = 0.3y$ (ii) $y' = 0.3t$

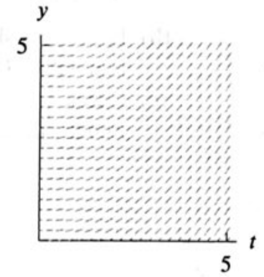
(b) For the solutions to each differential equation in part (a), what is $\lim_{t\to\infty} y(t)$ if

(i) $y(0) = 1$ (ii) $y(0) = 0$

(A)



(B)



14. A slope field for a differential equation has slopes greater than 0 at every point of the xy-plane.

(a) Explain why any solution curve to this differential equation is increasing everywhere.

(b) Can you conclude that a solution curve to this differential equation is concave up everywhere?

15. Sketch a slope field with the following properties. (Draw at least ten line segments, including some with $x < 0$, with $x > 0$, and with $x = 0$.)

$$\frac{dy}{dx} > 0 \text{ for } x < 0,$$
$$\frac{dy}{dx} < 0 \text{ for } x > 0,$$
$$\frac{dy}{dx} = 0 \text{ for } x = 0.$$

16. Sketch a slope field with the following properties. (Draw at least ten line segments, including some with $P < 2$, with $2 < P < 5$, and with $P > 5$.)

$$\frac{dP}{dt} > 0 \text{ for } 2 < P < 5,$$
$$\frac{dP}{dt} < 0 \text{ for } P < 2 \text{ or } P > 5,$$
$$\frac{dP}{dt} = 0 \text{ for } P = 2 \text{ and } P = 5.$$

17. Is the solution curve to $y' = 2x - 3y - 1$ concave up or concave down at $(3, 2)$?

18. **(a)** Sketch the slope field for the equation $y' = x - y$ in Figure 11.21 at the points indicated.

(b) Find the equation for the solution that passes through the point $(1, 0)$.

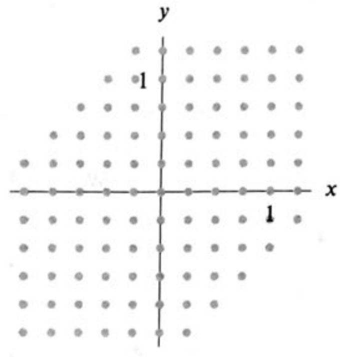


Figure 11.21

19. The slope field for the equation $dP/dt = 0.1P(10-P)$, for $P \geq 0$, is in Figure 11.22.

(a) Sketch the solutions that pass through the points

(i) $(0, 0)$ (ii) $(1, 4)$ (iii) $(4, 1)$
(iv) $(-5, 1)$ (v) $(-2, 12)$ (vi) $(-2, 10)$

(b) For which positive values of P are the solutions increasing? Decreasing? If $P(0) = 5$, what is the limiting value of P as t gets large?

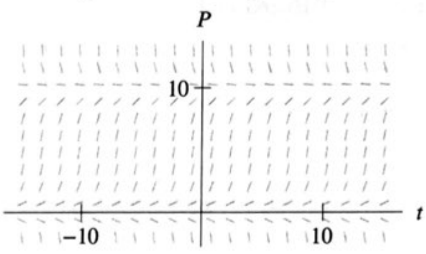


Figure 11.22

20. The slope field for $y' = 0.5(1 + y)(2 - y)$ is shown in Figure 11.23.

(a) Plot the following points on the slope field:

(i) the origin (ii) $(0, 1)$ (iii) $(1, 0)$
(iv) $(0, -1)$ (v) $(0, -5/2)$ (vi) $(0, 5/2)$

(b) Plot solution curves through the points in part (a).

(c) For which regions are all solution curves increasing? For which regions are all solution curves decreasing? When can the solution curves have horizontal tangents? Explain why, using both the slope field and the differential equation.

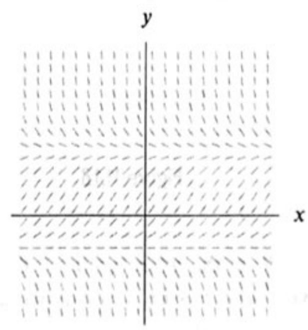


Figure 11.23: Note: x and y scales are equal

21. One of the slope fields in Figure 11.24 is the slope field for $y' = (x+y)/(x-y)$. Which one?

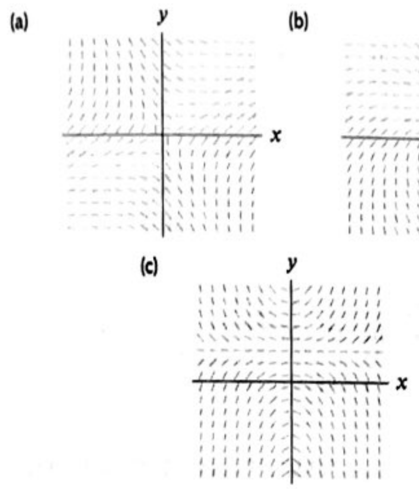


Figure 11.24

22. The slope field for $y' = (\sin x)(\sin y)$ is in Figure 11.25.

(a) Sketch the solutions that pass through the points

(i) $(0, -2)$ (ii) $(0, \pi)$

(b) What is the equation of the solution that passes through $(0, n\pi)$, where n is any integer?

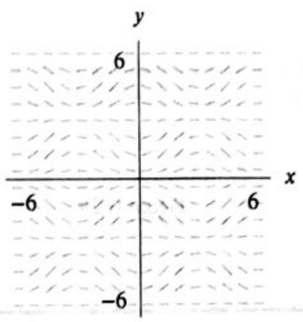


Figure 11.25

23. Match the slope fields in Figure 11.26 with their differential equations. Explain your reasoning.

(a) $y' = -y$ **(b)** $y' = y$ **(c)** $y' = x$

(d) $y' = 1/y$ **(e)** $y' = y^2$

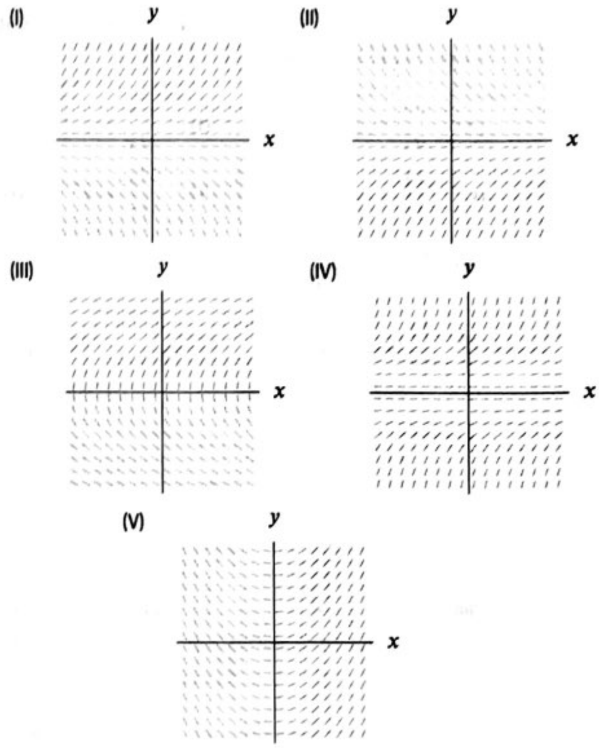


Figure 11.26: Each slope field is graphed for $-5 \le x \le 5, -5 \le y \le 5$

24. Match the slope fields in Figure 11.27 to the corresponding differential equations:

(a) $y' = xe^{-x}$ **(b)** $y' = \sin x$ **(c)** $y' = \cos x$

(d) $y' = x^2e^{-x}$ **(e)** $y' = e^{-x^2}$ **(f)** $y' = e^{-x}$

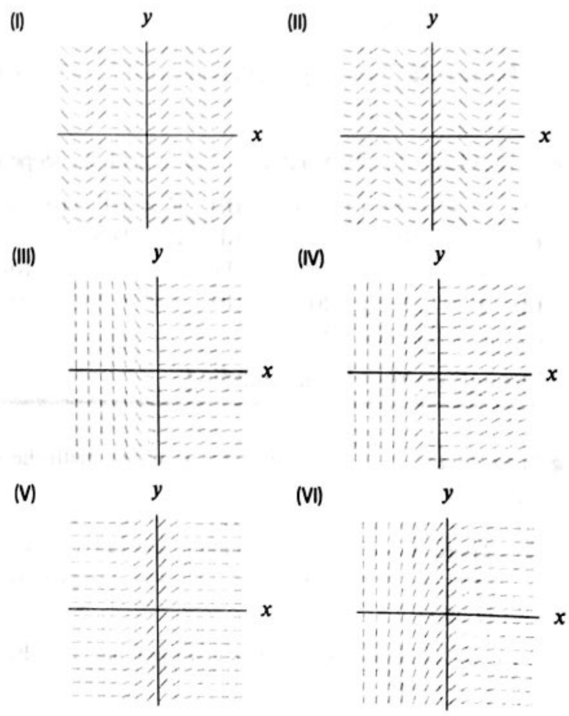


Figure 11.27

25. Match the following differential equations with the slope fields shown in Figure 11.28.

(a) $\dfrac{dy}{dx} = e^{x^2}$ (b) $\dfrac{dy}{dx} = e^{-2x^2}$

(c) $\dfrac{dy}{dx} = e^{-x^2/2}$ (d) $\dfrac{dy}{dx} = e^{-0.5x}\cos x$

(e) $\dfrac{dy}{dx} = \dfrac{1}{(1+0.5\cos x)^2}$

(f) $\dfrac{dy}{dx} = -\dfrac{e^{-x^2}}{y}$

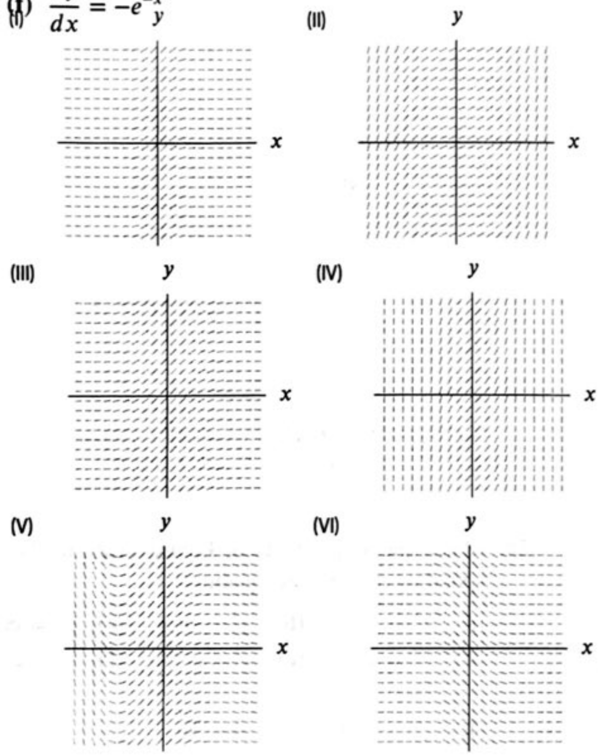
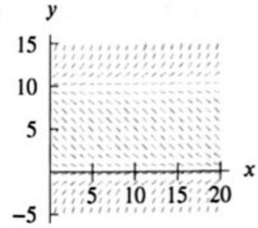


Figure 11.28: Each slope field is graphed for $-3 \le x \le 3$, $-3 \le y \le 3$

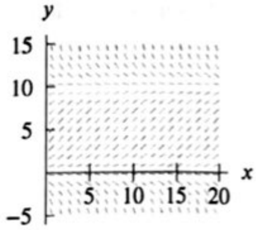
■ In Problems 26–29, match an equation with the slope field.

(a) $y' = 0.05y(10-y)$ (b) $y' = 0.05x(10-x)$
(c) $y' = 0.05y(5-y)$ (d) $y' = 0.05x(5-x)$
(e) $y' = 0.05y(y-10)$ (f) $y' = 0.05x(x-10)$
(g) $y' = 0.05y(y-5)$ (h) $y' = 0.05x(x-5)$
(i) $y' = 0.05x(y-5)$

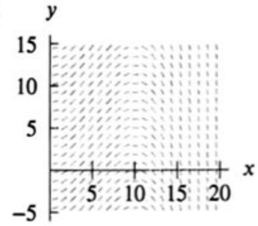
26.



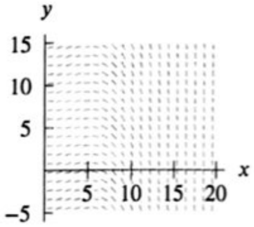
27.



28.



29.



30. Match the slope fields in Figure 11.29 to the differential equations; find a and b assuming $a \neq b$.

(a) $\dfrac{dy}{dx} = (x-a)(y-b)$ (b) $\dfrac{dy}{dx} = \dfrac{1}{(x-a)(y-b)}$

(c) $\dfrac{dy}{dx} = \dfrac{x-a}{y-b}$ (d) $\dfrac{dy}{dx} = \dfrac{y-b}{x-a}$

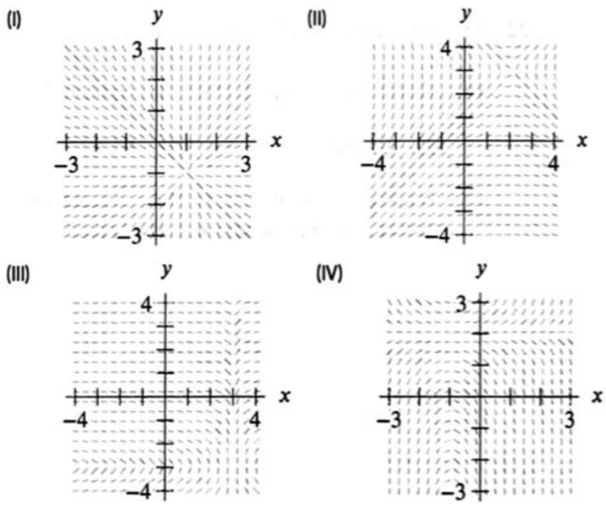


Figure 11.29

Strengthen Your Understanding

■ In Problems 31–33, explain what is wrong with the statement.

31. There is a differential equation that has $y = x$ as one of its solutions and a slope field with a slope of 0 at the point $(1, 1)$.

32. The differential equation $dy/dx = 0$ has only the solution $y = 0$.

33. Figure 11.30 shows the slope field of $y' = y$.

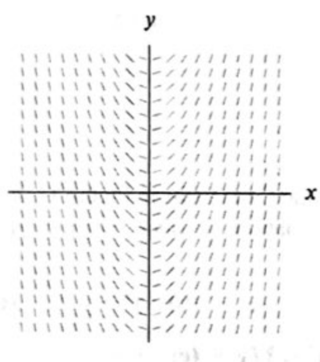


Figure 11.30