Math 116 — Practice for Exam 1

Generated September 6, 2017

Name: SOLUTIONS

Instructor: ______________________ Section Number: __________

1. This exam has 4 questions. Note that the problems are not of equal difficulty, so you may want to skip over and return to a problem on which you are stuck.
2. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you hand in the exam.
3. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.
4. Show an appropriate amount of work (including appropriate explanation) for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.
5. You may use any calculator except a TI-92 (or other calculator with a full alphanumeric keypad). However, you must show work for any calculation which we have learned how to do in this course. You are also allowed two sides of a $3'' \times 5''$ note card.
6. If you use graphs or tables to obtain an answer, be certain to include an explanation and sketch of the graph, and to write out the entries of the table that you use.
7. You must use the methods learned in this course to solve all problems.

Semester	Exam	Problem	Name	Points	Score
Winter 2010	1	2	ice cream	13	
Fall 2011	1	2	photo sharing	12	
Winter 2012	1	3	island	12	
Fall 2006	1	3	cookies	12	
Total				49	

Recommended time (based on points): 44 minutes

2. [13 points] An ice cream cone has a height of 15 centimeters and the diameter of the top is 5 centimeters. The cone is filled with soft-serve ice cream such that the ice cream completely fills the cone, but does not exceed the top of the cone. The ice cream has a constant density of 2 grams per cubic centimeter.

a. [5 points] Write an expression for the approximate mass of ice cream contained in a circular cross-sectional slice that is located h_i centimeters from the the bottom tip of the cone and has depth Δh centimeters . Your answer may be in terms of h_i and Δh. Don't forget to include units.

Solution: Using similar triangles, at h_i centimeters from the tip of the cone, the radius is $\frac{1}{6}h_i$. This means the volume of a slice of depth Δh at height h_i is approximately $\frac{\pi}{36}h_i^2\Delta h$. Therefore, the mass of the slice is volume times density, which gives $\frac{2\pi}{36}h_i^2\Delta h = \frac{\pi}{18}h_i^2\Delta h$ grams.

b. [4 points] Set up a definite integral that can be used to determine the EXACT total mass of ice cream that is filling the cone, then solve for this exact value. Include appropriate units in your answer.

Solution: By summing up all such slices as found in part (a) and letting $\Delta h \to 0$, we have

$$\text{Total Mass} = \int_0^{15} \frac{\pi}{18}h^2 dh = \frac{\pi}{54}h^3|_0^{15} = \frac{3375\pi}{54} = \frac{125\pi}{2}\text{grams.}$$

c. [4 points] At what height above the tip of the cone is the center of mass of the ice cream? Give an EXACT answer, show all work, and include appropriate units.

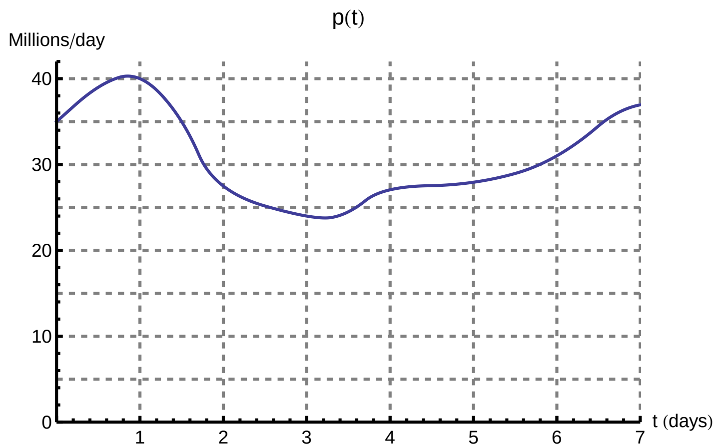
Solution: We already found the total mass in part (b), but we need to calculate the total moments. We have

$$\int_0^{15} \frac{\pi}{18}h^3 dh = \frac{\pi}{72}h^4|_0^{15} = \frac{50625\pi}{72} = \frac{5625\pi}{8}.$$

Putting this together, we get that

$$\overline{h} = \frac{\frac{5625\pi}{8}}{\frac{125\pi}{2}} = \frac{45}{4} \text{ cm from the bottom of the cone.}$$

2. [12 points] Photo sharing through social networking sites has become increasingly popular over the years. Suppose $p(t)$ gives the rate at which photos are uploaded to Facebook's servers, over a certain one-week period, in millions of photos per day. ($t = 0$ corresponds to the beginning of Sunday.) A graph of $p(t)$ is given below.



a. [2 points] Write a definite integral that gives the total number of photos uploaded to Facebook from the beginning of Sunday through the end of Monday. Include units in your answer.

Solution: $\displaystyle\int_0^2 p(t)dt$ millions of photos *or* $10^6 \displaystyle\int_0^2 p(t)dt$ photos

b. [8 points] Estimate the value of the definite integral in part (a) using LEFT(2), RIGHT(2), MID(2) and TRAP(2). Write each sum in terms of p.

Solution: $\Delta t = \frac{2-0}{2} = 1$, so the width of each rectangle in the Riemann sums is 1. Then

$$\text{LEFT}(2) = p(0)\Delta t + p(1)\Delta t \approx 35 \cdot 1 + 40 \cdot 1 = 75 \text{ million photos}$$

$$\text{RIGHT}(2) = p(1)\Delta t + p(2)\Delta t = 40 \cdot 1 + 27.5 \cdot 1 \approx 67.5 \text{ million photos}$$

$$\text{MID}(2) = p(0.5)\Delta t + p(1.5)\Delta t = 38 \cdot 1 + 35 \cdot 1 \approx 73 \text{ million photos}$$

$$\text{TRAP}(2) = \frac{\text{LEFT}(2) + \text{RIGHT}(2)}{2} \approx \frac{75 + 67.5}{2} = 71.25 \text{ million photos}$$

In terms of the function p, one can also calculate

$$\text{TRAP}(2) = \frac{1}{2}p(0) + p(1) + \frac{1}{2}p(2) \approx \frac{1}{2} \cdot 35 + 40 + \frac{1}{2} \cdot 27.5 = 71.25 \text{ million photos.}$$

Note that answers may vary slightly because exact values of the $p(t)$ were not given in the statement of the problem.

c. [2 points] Give a real world interpretation of the quantity $\frac{1}{5}\int_1^6 p(t)dt$. Include units.

Solution: Since $\frac{1}{5}\int_1^6 p(t)dt = \frac{1}{6-1}\int_1^6 p(t)$, the given integral represents the average rate at which photos are uploaded to Facebook's servers during the work week (Monday through Friday).

3. [12 points] A boat travels in a straight line toward an island d km away. The velocity $v(t)$ (toward the island is positive velocity) in km/hr, t hours after departing from its starting position. The velocity $v(t)$ during the first three hours is recorded at half hour intervals, and is given in the table below:

t	0	0.5	1	1.5	2	2.5	3
$v(t)$	50	48	44	38	30	20	8

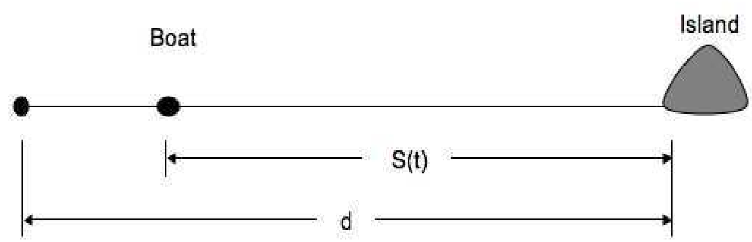
a. [8 points] Find an estimate for how far the boat is from the starting point after 3 hours using the four approximations LEFT, RIGHT, MID and TRAP. Use the maximum number of subintervals possible. Write each sum, and justify whether the sum is an underestimate or an overestimate. Assume the velocity is always decreasing and has no inflection points. **Circle your answers**.

Solution:

$$\begin{aligned} \text{LEFT}(6) &= \frac{1}{2}(50+48+44+38+30+20) = 115 && \text{overestimate.} \\ \text{RIGHT}(6) &= \frac{1}{2}(48+44+38+30+20+8) = 94 && \text{underestimate.} \\ \text{TRAP}(6) &= \frac{1}{2}(115+94) = 104.5 && \text{underestimate.} \\ \text{MID}(3) &= (48+38+20) = 106 && \text{overestimate.} \end{aligned}$$

since $v(t)$ is decreasing and concave down.

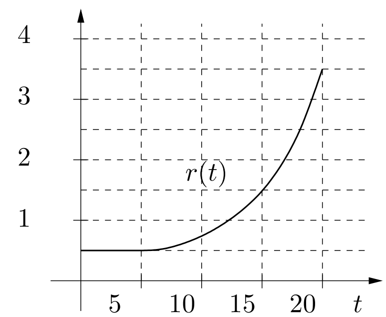
b. [4 points] If it takes the boat 5 hours to reach the island, write an expression involving integrals for the distance $S(t)$ between the island and the boat for $0 \le t \le 5$.



Solution:

$$S(t) = d - \int_0^t v(x)dx.$$

3. [12 points] Having completed their team homework, Alex and Chris are making chocolate chip cookies to celebrate. The rate at which they make their cookies, $r(t)$, is given in cookies/minute in the figure to the right (in which t is given in minutes). After $t =$ 20 minutes they have completed their cookie making extravaganza.



(a) [3 of 12 points] Write an expression for the total number of cookies that they make in the 20 minutes they are baking. Why does your expression give the total number of cookies?

Solution:
We are given the rate at which the cookies are being produced, so we know by the Fundamental Theorem of Calculus that the total number of cookies produced is given by $\int_0^{20} r(t)\,dt$.

(b) [3 of 12 points] Using $\Delta t = 5$, find left- and right-Riemann sum and trapezoid estimates for the total number of cookies that they make.

Solution:
Using $\Delta t = 5$, the left- and right-hand Riemann sums are

$$\text{LEFT}(4) \approx 5(0.5 + 0.5 + 0.75 + 1.5) = 16.25$$
$$\text{RIGHT}(4) \approx 5(0.5 + 0.75 + 1.5 + 3.5) = 31.25.$$

Thus the trapezoid estimate is $\text{TRAP}(4) = \frac{1}{2}(16.25 + 31.25) = 23.75$, or about 24 cookies.

(c) [3 of 12 points] How large could the error in each of your estimates in (b) be?

Solution:
We know that the maximum error in the left- or right-hand sums is just $\Delta t(r(20) - r(0)) = 5(3.5 - 0.5) = 15$ cookies. The maximum error in the trapezoid estimate is half this, or 7.5 cookies.

(d) [3 of 12 points] How would you have to change the way you found each of your estimates to reduce the possible errors noted in (c) to one quarter of their current values?

Solution:
The error in the left- or right-hand sums drops as n, so we would have to take four times as many steps, reducing Δt to $\frac{5}{4} = 1.25$ min to reduce the error in those estimates by a factor of four. The error in the trapezoid estimate drops as n^2, the square of the number of steps we take in the calculation, so if we recalculated our estimate with $\Delta t = 2.5$ minutes it would be four times as accurate.