INFINITE DIMENSIONAL EQUIVARIANT COMMUTATIVE ALGEBRA

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1. Introduction

(A) Let $R = \mathbb{C}[x_1, x_2, \ldots]$ be the polynomial ring in infinitely many variables. Consider the group action $\mathfrak{S}_\infty = \bigcup_{n\geqslant 1} \mathfrak{S}_n$. We have following theorem

Theorem 1.1 (Cohen, 1967)**.** *R is $\mathfrak{S}_\infty$-notherian, i.e., ACC holds for $\mathfrak{S}_\infty$-ideals.*

Applications:
- Cohen: any variety of metablien groups is finitely based
- Draisma-Eggermont: $\mathrm{Gr}_p(V) \subseteq \mathbb{P}(\wedge^p V) \rightsquigarrow \mathrm{Sec}_k(\mathrm{Gr}_p(V))$. Fix k, there exists d such that $\mathrm{Sec}_k(\mathrm{Gr}_p(V))$ are cut out by equations of degree $\leqslant d$ for any p and V.
- Draisma-Kuttar: similar result for Segre embeddings

(B) $\mathrm{GL}_\infty = \bigcup_{n\geqslant 1} \mathrm{GL}_n$

Definition 1.2. A representation of GL_∞ is *polynomial* if it appears in a sum of tensor powers of the standard representation $\mathbb{C}^\infty = \cup \mathbb{C}^n$.

Example 1.3.
- $(\mathbb{C}^\infty)^{\otimes h}$.
- $\mathrm{Sym}^k(\mathbb{C}^\infty)$
- $\bigwedge^k(\mathbb{C}^\infty)$
- $S_\lambda(\mathbb{C}^\infty)$ where λ is a partition

Definition 1.4. A GL-algebra is a $\mathbb{C}$-algebra R equipped with an action of GL_∞ such that it forms a polynomial representation.

Example 1.5.
$R = \mathbb{C}[x_1, x_2, \ldots] = \oplus_{k>0} \mathrm{Sym}^k(\mathbb{C}^\infty)$
$R = \mathbb{C}[x_i, y_i, z_i]_{i\geqslant 1} = \mathrm{Sym}(\mathbb{C}^\infty \oplus \mathbb{C}^\infty \oplus \mathbb{C}^\infty)$
$R = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty) = \mathbb{C}[x_{i,j}]_{1\leqslant i\leqslant j<\infty}$.
$R = \mathbb{C}[x_i, y_j]/(x_i y_j - x_j y_i)$.

Theorem 1.6 (Drasma)**.** *Let $R = \mathrm{Sym}(V)$ where V is some finite length polynomial representation. Then ACC holds for* radical GL*-ideals.*

Remark 1.7. ACC for arbitrary GL-ideals is unknown, which is a very important problem.

Applications
- $R = \mathrm{Sym}(\mathrm{Sym}^2(\mathbb{C}^\infty) \oplus \mathrm{Sym}^3(\mathbb{C}^\infty))$. Then $\mathrm{Spec}(R) = (\mathrm{Sym}^2)^* \times (\mathrm{Sym}^3)^*$ is the space of pairs (f, g) where f is homogeneous degree 2 polynomials and g is homogeneous degree 3 polynomials. Both are in infinitely many variables.
- Erman-Sum-S: proved Stillman's conjecture using this approach.
- Draisma-Lasón-Leykm: finiteness properties of Gröbner basis.

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2. Equivalent Commutative Algebra

Fix a commutative ring R on which a group G acts. The goal is to import definitions from commutative algebra to equivariant setting.

The principal is that we want to phrase things using ideals (no elements) then change "ideal" to "G-ideal".

Definition 2.1 (Classical)**.** $\mathfrak{p}$ is a prime ideal if $xy \in \mathfrak{p}$ then $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Definition 2.2 (Equivariant)**.** A G ideal $\mathfrak{p}$ is a G-prime ideal if $\mathfrak{a}\mathfrak{b} \subseteq \mathfrak{p}$ then $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$ for any G-ideals $\mathfrak{a}, \mathfrak{b}$.

This is also an element form:

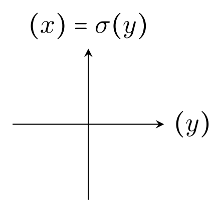
Definition 2.3. $\mathfrak{p}$ is a G-prime if $x \cdot gy \in \mathfrak{p}$ for any $g \in G$, then either $x \in \mathfrak{p}$ or $y \in \mathfrak{p}$.

Example 2.4. Take $R = \mathbb{C}[x, y]$ and $G = \mathfrak{S}_2$, $\mathfrak{p} = (xy)$.

Claim. *$\mathfrak{p}$ is a G-prime.*

Proof. Given that $f \cdot \sigma y \in \mathfrak{p}$ for any $\sigma \in \mathfrak{S}_2$. Suppose that $f \notin \mathfrak{p}$, say $x \nmid f$, then $xy | f \cdot \sigma g \Rightarrow x | \sigma g$ for any σ, which implies that $x | g, y | g$. So $xy | g \Rightarrow g \in \mathfrak{p}$. □

$V(\mathfrak{p}) =$



Fact 2.5. *Suppose that G is finite, R noetherian. Then*

- *if $\mathfrak{p}$ is a prime of R, then $\bigcap_{\sigma \in G} \sigma(\mathfrak{p})$ is G-prime.*
- *there is a bijection $\{primes\}/G \cong \{G\text{-}primes\}$.*

Example 2.6. Let $R = \mathbb{C}[x_i]_{i \geqslant 1}$, $G = \mathfrak{S}_\infty$ and $\mathfrak{p} = (x_i^2)_{i \geqslant 1}$.

Claim. *$\mathfrak{p}$ is G-prime*

Proof. Say $f \cdot \sigma(g) \in \mathfrak{p}$ for any $\sigma \in G$. Choose σ such that f and g have no variables in common. Then f or g belongs to $\mathfrak{p}$, which implies that $\mathfrak{p}$ is G-prime. □

Definition 2.7 (Classical)**.** The radical of an ideal $\mathfrak{a}$ is the set of elements x such that $x^n \in \mathfrak{a}$ for some n. So $\mathrm{Rad}(\mathfrak{a})$ =sum of all ideals $\mathfrak{b}$ such that $\mathfrak{b}^n \subseteq \mathfrak{a}$ for some n.

Definition 2.8 (Equivariant)**.** The G-radical of a G-ideal $\mathfrak{a}$, denoted $\mathrm{Rad}_G(\mathfrak{a})$, is the sum of all G-ideals $\mathfrak{b}$ such that $\mathfrak{b}^n \subseteq \mathfrak{a}$ for some n.

Fact 2.9. $\mathrm{Rad}_G(\mathfrak{a}) = \bigcap_{\mathfrak{a} \subseteq \mathfrak{p}} \mathfrak{p}$ *where $\mathfrak{p}$ runs through all G-primes.*

Definition 2.10. The G-spectrum of R, $\mathrm{Spec}_G(R)$, is the set of all G-primes, with Zariski topology.

Remark 2.11. If $\mathfrak{a}, \mathfrak{b}$ are G-ideals, then $V_G(\mathfrak{a}) = V_G(\mathfrak{b})$ if and only if $\mathrm{Rad}_G(\mathfrak{a}) = \mathrm{Rad}_G(\mathfrak{b})$.

The main problem is to understand Spec_G in examples (A) and (B).

3. Example A

(Joint with Rohit Nagpel) Let $R = \mathbb{C}[x_1, x_2, \ldots]$ and $G = \mathfrak{S}_\infty$. To start, let's consider radical G-primes

$$X = \operatorname{Spec}(R) = \mathbb{C}^\infty = \{(a_1, a_2, a_3, \ldots) | a_i \in \mathbb{C}\}$$

radical G-primes $\Leftrightarrow$ G-irreducible closed sets of X.

Construction 3.1 (Bound number of values). $Z_1 \subseteq \mathbb{C}^\infty$ consisting of tuples use at most 2 values, e.g., $(a, a, a, b, b, b, b, a, \ldots)$.

Claim. *Z_1 is Zariski closed*

Defined by vanishing of 3-variable discriminants $(x_i - x_j)(x_i - x_k)(x_j - x_k) = 0$ for any i, j, k.

Construction 3.2 (Bound multiplicities of values). $Z_2 \subseteq Z_1$ is the set where b appears at most once. This is defined by $(x_i - x_j)(x_k - x_l) = 0$ for any i, j, k, l distinct.

Construction 3.3 (Algebraic relations between values). $Z_3 \subseteq Z_2$ is the set where $f(a, b) = 0$ for $f \in \mathbb{C}[x, y]$. This is defined by $(x_i - x_j)(x_i - x_k) f(x_j, x_i) = 0$.

Construction 3.4 (General construction).

- A partition of ∞ is a tuple $\lambda = (\lambda_1, \cdots, \lambda_r)$ where $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ where $\lambda_i \in \mathbb{N} \cup \{\infty\}$ and $\lambda_1 = \infty$.
- $X_\lambda \subseteq X$ consisting of points of type λ ($Z_2 = X_{(\infty,1)}$)
- $\varphi : X_\lambda / \mathfrak{S}_\infty \cong \mathbb{A}^{[r]} / \operatorname{Aut}(\lambda)$ where $\mathbb{A}^{[r]} \subseteq \mathbb{A}^r$ are the points where coordinates are distinct and $\operatorname{Aut}(\lambda) \subseteq \mathfrak{S}_r$ fixes λ.
- Given a closed subset Z of RHS above, denote $X_\lambda(Z) \subseteq X$ be the Zariski closure of $\varphi^{-1}(Z)$.

Theorem 3.5 (Nagpel-S.)**.** *Have bijections*

$$\{\textit{proper } G\textit{-irreducible subsets of } X\} \leftrightarrow \{(\lambda, Z) | \lambda \textit{ partition of } \infty, Z \textit{ as above, irreducible}\}$$
$$X_\lambda(Z) \leftrightarrow (\lambda, Z)$$

$$\{G\text{-primes of } R\} \leftrightarrow \{\text{radical } G\text{-primes}\}$$

Goal is to understand fibers.

Let $\mathfrak{q} = \langle x_i - x_j \rangle_{i,j \geqslant 1} \subseteq R$ be a radical G-prime. It is sufficient to understand fibers over $\mathfrak{q}$.

We define $\mathfrak{q}_n = \langle (x_i - x_j)^n \rangle_{i,j \geqslant 1} \subseteq R$.

Theorem 3.6 (Nagpel-S)**.**

- *$\mathfrak{q}_n$ is G-prime if and only if n is odd.*
- *The $\mathfrak{q}_n$ with n odd are exactly G-primes with radical $\mathfrak{q}$.*

The key ingredient is that

$$\begin{aligned}
A &= \mathbb{C}[x_1, \ldots, x_n] \\
B &= \mathbb{C}[x_i - x_j] \subseteq A \\
&= \mathbb{C}[y_2, \ldots, y_n] \quad \text{where } y_i = x_i - x_1 \\
I &= \langle x_1^k, \ldots, x_n^k \rangle \\
J &= B \cap I \text{ contraction}
\end{aligned}$$

$$(y_i - y_j)^{2k-1} = (x_i - x_j)^{2k-1} = \sum_{r=0}^{2k-1} \binom{2k-1}{r} x_i^r x_j^{2k-1-r}$$

Since at least one of the exponent on RHS is $\leqslant k$, we conclude that $(y_i - y_j)^{2k-1} \in J$.

Theorem 3.7. $J = \langle (y_i - y_j)^{2k-1} \rangle_{2 \leqslant i,j, \leqslant n}$.

4. Example B

Let R be a G-algebra where $G = \mathrm{GL}$. Goal is to describe $\mathrm{Spec}_G(R)$.

Example 4.1.

- $R = \mathrm{Sym}(\overbrace{\mathbb{C}^\infty \oplus \cdots \oplus \mathbb{C}^\infty}^{d \text{ copies}})$

  Proposition 4.2 (Sam-S.)**.** *G-primes are G-stables primes.*

  So $\mathrm{Spec}_G(R) = \coprod_{r=0}^{d} \mathrm{Gr}_r(\mathbb{C}^d)$ (topology is *not* disjoint union).
- $R = \oplus_{n=0}^{\infty} \bigwedge^{2n}(\mathbb{C}^\infty)$. (0) is G-prime, i.e., R = GL-domain. $\mathfrak{a}, \mathfrak{b} \subseteq R$ nonzero G-ideals, need to show that $\mathfrak{a}\mathfrak{b} \neq (0)$.
- $R = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty) = \oplus_{\lambda \text{ even}} S_\lambda(\mathbb{C}^\infty)$.
  - $\mathfrak{p}_r$ is the determinantal ideal, G-stable prime.
  - $\mathfrak{p}_{r,s}$ ideal geneated by $(2s+2) \times (r+1)$ rectangle.

  There are G-prime.

For $R = \mathrm{Sym}(\mathrm{Sym}^2)$, $R(\mathbb{C}^\infty) = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty)$ and $R(\mathbb{C}^{\infty|\infty}) = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^{\infty|\infty}) = \mathrm{Sym}(\mathrm{Sym}^2 \mathbb{C}^\infty \oplus \bigwedge^2 \mathbb{C}^\infty[z] \oplus \underbrace{\text{cross terms}}_{\text{nilpotent}})$

$V(\mathfrak{p}_{r,s}(\mathbb{C}^{\infty|\infty}))$ =rank s, r on even part, rank $\leqslant s$ on odd part.

Theorem 4.3. *R is a* GL*-algebra, $\mathfrak{a}, \mathfrak{b} \subseteq R$ are* GL*-ideals. Then*

$$\mathrm{Rad}_G(\mathfrak{a}) = \mathrm{Rad}_G(\mathfrak{b}) \Leftrightarrow V(\mathfrak{a}(\mathbb{C}^{\infty|\infty})) = V(\mathfrak{b}(\mathbb{C}^{\infty|\infty}))$$