COHEN-MACAULAYNESS OF ABSOLUTE INTEGRAL CLOSURES

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1. The Hochster-Huneke Theorem

Recall: $(R, \mathfrak{m})$ a noetherian local ring. It is Cohen-Macaulay if every system of parameters is a regular sequence, or equivalently, $\mathrm{H}^i_{\mathfrak{m}}(R) = 0$ for all $i < \dim(R)$.

Theorem 1.1 (Hochster-Huneke)**.** *Let $(R, \mathfrak{m})$ be an excellent noetherian local domain over $\mathbb{F}_p$. Let R^+ be the absolute integral closure of R. Then R^+ is CM.*

Remark 1.2. (1) The Theorem 1.1 fails in characteristic 0 (containing $\mathbb{Q}$) if $\dim(R) \geqslant 3$.
(2) There is a graded analogue of Theorem 1.1 (Projective version).

Theorem 1.3 (HH-graded)**.** *Let k be a field of characteristic p, X/k projective variety, let $\mathcal{L} \in \mathrm{Pic}(X)$ be an ample line bundle. Then there exists a finite surjective map $\pi : Y \to X$ such that π^* kills the following:*
(a) $\mathrm{H}^{>0}(X, \mathcal{O}_X)$*;*
(b) $\mathrm{H}^{>0}(X, \mathcal{L})$*;*
(c) $\mathrm{H}^{<d}(X, \mathcal{L}^{-1})$ *where* $d = \dim(X)$*.*

(3) Theorem 1.1 is equivalent to that if R is a regular excellent local domain, then $R \to R^+$ is faithfully flat.

Some applications of Theorem 1.1:

(1) Hochster-Roberts theorem: $R \hookrightarrow S$ is a direct summand as R-modules. If S is regular over $\mathbb{F}_p$, then R is Cohen-Macaulay.
(2) Direct summand conjecture: If $R \hookrightarrow S$ is a finite injective map, R regular over $\mathbb{F}_p$, then $R \hookrightarrow S$ is a direct summand.
(3) Faltings connected theorem: Let $(R, \mathfrak{m})$ be an excellent local domain over $\mathbb{F}_p$. Let $I = (x_1, \ldots, x_n)R \subseteq \mathfrak{m}$ where $n \leqslant \dim(R) - 2$. Then $\mathrm{Spec}(R/I) \setminus \{\mathfrak{m}\}$ is connected.
(4) Kollár's conjecture on local Picard groups (B-dJ).

The goal of the rest talk is to explain analogue of Theorem 1.1 in mixed characteristic.

2. Statements

Fix a prime p.

Theorem 2.1 (joint with Luire)**.** *Let $(R, \mathfrak{m})$ be an excellent noetherian local domain, $p \in \mathfrak{m}$. Fix $R \to R^+$ an absolute integral closure. Then R^+ is* almost CM, *i.e.,* $p^{1/p^n} \cdot \mathrm{H}^i_{\mathfrak{m}}(R^+) = 0$ *for all* $n \geqslant 0, i < \dim(R)$*.*

Remark 2.2. (1) This generalizes a result of Heitmann: Heitmann proved Theorem 2.1 for $\dim(R) \leqslant 3$ and hence the direct summand conjecture in dim 3.
(2) Get a new prove of DSC (André): The Riemann extension theorem is not used here.

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Theorem 2.3 (joint with Lurie)**.** *Say R is a noetherian ring, X/R is a proper R-scheme. Then there exists $\pi : Y \to X$ finite surjective map such that*

$$\mathrm{H}^{>0}(X, \mathcal{O}_X) \to \mathrm{H}^{>0}(Y, \mathcal{O}_Y)$$

is divisible by p as a map.

Remark 2.4.

(1) Theorem 2.3 implies that spliter$[\frac{1}{p}]$ have rational singularities.
(2) Theorem 2.3 is previous known if R has characteristic p or $\dim(R) = 1$.

Theorem 2.5 (Main Theorem)**.**

(1) Let $(R, \mathfrak{m})$ be an excellent noetherian local domain and $p \in \mathfrak{m}$. Then R^+ is CM (in the sense of local cohomology).
(2) Let R be any commutative ring, X/R a proper R-scheme. Let $\mathcal{L} \in \mathrm{Pic}(X)$ semiample and big. Then there exists a finite cover $\pi : Y \to X$ such that

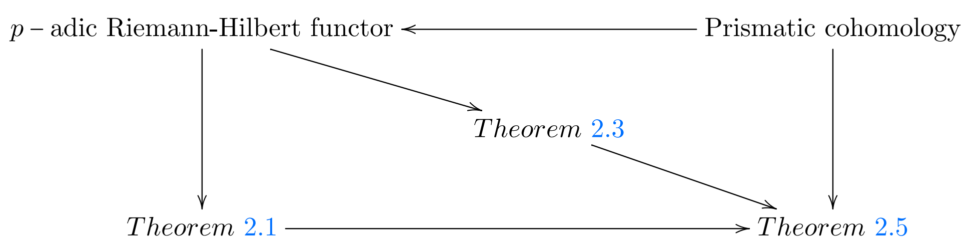
$$\mathrm{H}^*(X, \mathcal{L}^{-1})_{p\text{–torsion}} \to \mathrm{H}^*(Y, \mathcal{L}^{-1})_{p\text{–torsion}}$$

is 0.

Consequences: fix $(R, \mathfrak{m})$ as above.

(1) If R is a splinter, then R is CM.
(2) $R^+/p^{1/p^\infty} R^+$ is CM over R/p (in any sense). (Lyubeznik's question)
(3) Get a new/explicit constructions of weakly functorial big CM algebras: Let $(R, \mathfrak{m})$ be a complete noetherian local domain, $p \in \mathfrak{m}$. Then $\widehat{R}^+$ is CM over R (in all senses).

Architecture of the proof:



3. The Huneke-Lyubeznik approach to Theorem 1.1

Theorem 3.1 (Huneke-Lyubeznik)**.** *Let $(R, \mathfrak{m})$ be an excellent noetherian local domain over $\mathbb{F}_p$ such that R has a dualizing complex. Then there exists a finite extension $R \to S$ such that $\mathrm{H}^i_{\mathfrak{m}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(S)$ is 0 for any $i < \dim(R)$.*

What goes into the point?

(1) Equational lemma: if $M \subseteq \mathrm{H}^i_{\mathfrak{m}}(R)$ is a finite length, Frobenius stable, R-submodule, then there exists a finite extension $R \to S$ such that $M \mapsto 0 \in \mathrm{H}^i_{\mathfrak{m}}(S)$.
(2) Induction on dimension: there exists a finite extension $R \hookrightarrow S$ such that $\mathrm{Im}(\mathrm{H}^i_{\mathfrak{m}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(S))$ has finite length for any $i < \dim(R)$.

Mystery: How to bring Frobenius in mixed char?

4. Prismatic cohomology

Setup: Let $V = \widehat{\overline{\mathbb{Z}_p}}$ is the p-adic completion of the integral closure of $\mathbb{Z}_p$ in $\overline{\mathbb{Q}_p}$. The fraction field of this ring is algebraically closed. Hence there is no arithmetic issue.

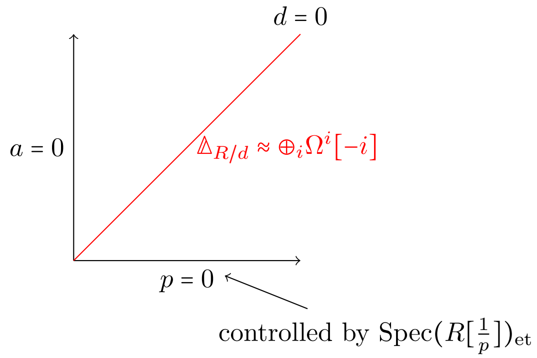
Let $A = \mathrm{Anf}(V) = \mathrm{W}((V/p)^\flat)$ carries a Frobenius automorphism φ. Then $A/(d) \cong V$ for some $d \in A$.

Previously there's no Frobenius action on V, but "one-parameter deformation" of V, i.e., A carries a Frobenius action.

Theorem 4.1. *Say R is a smooth algebra over V. Then there exists a natural commutative algebra object $\Delta_R \in D(A)$ and $\varphi : \Delta_R \to \Delta_R$ such that*

(1) *Hodge-Tate: Δ_R/d is closely related to R. (ex: the graded associated algebra of Δ_R/d is $\oplus_{i \geqslant 0} \Omega^i_{R/V}[-i]$.)*
(2) *Étale: Up to isogeny, Δ_R/p is controlled by $\mathrm{Spec}(R[\frac{1}{p}])_{\mathrm{et}}$. (ex: $\left(\Delta_{R/p}[\frac{1}{d}]\right)^{\varphi=1} = \mathrm{H}^*_{\mathrm{et}}(\mathrm{Spec}(R[\frac{1}{p}]), \mathbb{F}_p)$).*

Picture of Δ_R: write $d = a - p$



Key idea: Use $\Delta_{R/p}$ with Frobenius φ as a replacement for R with Frobenius φ in Theorem 3.1.

Remark 4.2. Geancvicus-Koshikawa extended Theorem 4.1 to the semistable case.

5. Outline of proof of Theorem 2.5

Setup: Let $R = \overline{\mathbb{Z}_p}[x_1, \ldots, x_n], \mathfrak{m} = (p, x_1, \ldots, x_n)$.

The goal is to prove $\mathrm{H}^i_{\mathfrak{m}}(R^+/p) = 0$ for any $i < n$.

Fix $\overline{K}$ = algebraic closure of $\mathrm{Frac}(R)$. Consider

$$P = \left\{ \begin{array}{ccc} \mathrm{Spec}(\overline{K}) & \longrightarrow & Y \\ & \searrow & \downarrow \text{alteration} \\ & & \mathrm{Spec}(R) \end{array} \quad \text{where } Y \text{ is semistable over } V \right\}$$

Observation: Theorem 2.3 + deJong $\Rightarrow \varinjlim_{Y \in P} \mathbb{R}\Gamma(Y, \mathcal{O}_Y)/p = R^+/p$.

$$\begin{aligned} \text{The goal} &\Leftrightarrow \varinjlim_{Y \in P} \mathrm{H}^i_{\mathfrak{m}}(\mathbb{R}\Gamma(Y, \mathcal{O}_Y)/p) = 0 \quad \forall i < n \\ &\Leftrightarrow \varinjlim_{Y \in P} \mathrm{H}^i_{\mathfrak{m}}(\Delta_Y/(p,d)) = 0 \quad \forall i < n \\ &\Leftrightarrow \varinjlim_{Y \in P} \mathrm{H}^i_{\mathfrak{m}}(\Delta_Y/p) = 0 \quad \forall i < n+1 \end{aligned}$$

Now argue as in Theorem 3.1:

(1) Equational lemma: finite length φ-stable submodules of $\mathrm{H}^i_{\mathfrak{m}}(\Delta_Y/p)$ can be killed by a map $Y' \to Y$.
(2) Induction on dim: for any $Y \in P$, there exists a map $Y' \to Y$ such that $\mathrm{Im}(\mathrm{H}^i_{\mathfrak{m}}(\Delta_Y/(p,d)) \to \mathrm{H}^i_{\mathfrak{m}}(\Delta_{Y'}/(p,d))$ is finite for $i < n$.
(3) Theorem 2.1$\Rightarrow$ for any $Y \in P$, there exists a map $Y' \to Y$ such that $\mathrm{Im}(\mathrm{H}^i_{\mathfrak{m}}(\Delta_Y/(p,d)) \to \mathrm{H}^i_{\mathfrak{m}}(\Delta_{Y'}/(p,d))$ is killed by d^c for some $c = c(n), i < n+1$.

All listed points above $\Rightarrow$ Theorem 2.5.