TWO APPLICATIONS OF p-DERIVATIONS TO COMMUTATIVE ALGEBRA

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1. Derivations

Definition 1.1. A *derivation* on a ring R is a map $\partial : R \to R$ such that

(1) $\partial(x + y) = \partial(x) + \partial(y)$
(2) $\partial(xy) = x\partial(y) + y\partial(x)$

for any $x, y \in R$.

(1) means that ∂ is $\mathbb{Z}$-linear.

Example 1.2. If $R = A[x_1, \dots, x_n]$ then $\frac{\partial}{\partial x_i} : R \to R$ "formal differentiation" is an A-linear derivation.

If A has characteristic 0, $f \in A[x_1, \dots, x_n]$, and $f \in (x_i^n)A \smallsetminus (x_i^{n+1})A$ for some $n > 0$, then $\frac{\partial f}{\partial x_i} \in (x_i^{n-1})A \smallsetminus (x_i^n)A$.

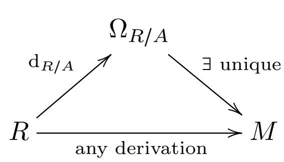
The conditions

(1) $\partial(x + y) = \partial(x) + \partial(y)$
(2) $\partial(xy) = x\partial(y) + y\partial(x)$

for any $x, y \in R$ are linear in ∂. So

- the set of A-linear derivations is an R-module by post-multiplication.
- can define $\partial : R \to M$ (R-module).

There is a universal object



So $\mathrm{Der}_{R/A}(M) \cong \mathrm{Hom}_R(\Omega_{R/A}, M)$.

The module $\Omega_{R/A}$ is called the module of Kähler differentials.

Example 1.3. (1) For $R = A[x_1, \dots, x_n]$, $\mathrm{Der}_{R/A}(R) = \oplus_{i=1}^n R\frac{\partial}{\partial x_i}$.
(2) For $k[x, y, z]/(x^2 + y^2 + z^2)$ where $\mathrm{char}(k) \neq 2$, the Kähler differential is given by

$$\mathrm{Der}_{R/k}(R) = R\langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}, y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\rangle$$

No erivations decrease degree or order; nothing like $\frac{\partial}{\partial x}$.

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This a notes of a talk in MSRI: Fellowship of the Ring by Professor Jack Jeffries, taken by Zhan Jiang, who is responsible for any and all errors. Please email zoeng@umich.edu with any corrections.

Theorem 1.4 (Nagata-Zariski-Lipman)**.** *If* $(R, \mathfrak{m})$ *has equal characteristic 0,* $x \in \mathfrak{m}$*,* $\partial \in \mathrm{Der}(R)$ *such that* $\partial(x) = 1$*, then* $\widehat{R} \cong R'[\![x]\!]$*.*

Here x is a formal regular parameter.

2. p-DERIVATIONS

Definition 2.1 (Joyal, Buivm)**.** Let p be a prime number, R a ring. A p-derivation on R is a map $\delta : R \to R$ such that

(0) $\delta(1) = 0$
(1) $\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}$.
(2) $\delta(xy) = x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y)$.

Since $i!(p-i)! | (p-1)!$ for $0 < i < p$, this is sensible in any characteristic.

Evidently, p-deriations are *not* $\mathbb{Z}$-linear/additive.

Moreover, these conditions are *not* linear in δ, so

- we cannot define a p-derivation $\delta : R \to M$ (into a module M).
- there is no analogue of Kähler differentials.

What do the axioms mean in Definition 2.1.

If $\delta : R \to R$ is a p-derivation, then $\Phi : R \to R$ given by $\Phi(x) = x^p + p\delta(x)$ is a ring homomorphism such that

$$\begin{array}{ccc} R & \xrightarrow{\Phi} & R \\ \downarrow & & \downarrow \\ R/pR & \xrightarrow{F} & R/pR \end{array}$$

commutes.

Such Φ is a *lift of Frobenius* (LOF).

If p is a regular element on R, then this yields a bijection between

$$\{p-\text{derivations}\} \Leftrightarrow \{\text{lifts of Frobenius}\}$$

We will mostly be focused on this case (mixed characteristic).

Example 2.2. (1) $R = \mathbb{Z}$, for any p, $\mathrm{id}_{\mathbb{Z}}$ is a unique LOF, so $\varphi_p(n) = \frac{n-n^p}{p}$ is the unique p-derivation, e.g., for $p = 3$, we have

$$1 \mapsto 0$$
$$2 \mapsto -2$$
$$3 \mapsto -8$$
$$4 \mapsto -20$$

(2) $R = \widehat{\mathbb{Z}}_{(p)}$ p-adic integers, id is the unique LOF, unique p-derivation by the same formula.
(3) (Strict p-rings) If (V, pV, k) is a complete DVR with uniformizer p and k perfect, then V admits unique LOF / p-derivation.
(4) Let A be a ring with p-derivation δ. Then for any $f_1, \ldots, f_n \in A[x_1, \ldots, x_n]$, there exists a unique p-derivation on $A[x_1, \ldots, x_n]$ such that $\widetilde{\delta}|_A = \delta$ and $\widetilde{\delta}(x_i) = f_i$. For $f_1 = \cdots = f_n = 0$, call it *standard*

extension or *standard p-derivation*. E.g., on $\mathbb{Z}[x_1, \dots, x_n]$, the standard derivation corresponds to LOF ψ with $\psi(x_i) = x_i^p$ for all i. So

$$\delta(f(\underline{x})) = \frac{f(\underline{x}^p) - f(\underline{x})^p}{p}$$

Freshman's nightmare.

Proposition 2.3. *Let R be a ring with p-derivation δ, p a regular element on R. Then for $f \in (p^n) \smallsetminus (p^{n+1})$ for some $n > 0$, $\delta(f) \in (p^{n-1}) \smallsetminus (p^n)$.*

Proof. Any p-derivation restricts to a p-derivation on $\mathbb{Z}$, so $\delta(p) = 1 - p^{p-1}$. If $f = pg$ where $g \in (p^{n-2}) \smallsetminus (p^n)$, then

$$\begin{aligned} \delta(f) = \delta(pg) &= p^p\delta(g) + g^p\delta(p) + p\delta(p)\delta(g) \\ &= p\delta(g) + g^p\delta(p). \end{aligned}$$

If $n = 1$, then the order ord_p of each term is

$$\underbrace{p}_{1}\,\delta(g) + \underbrace{g^p}_{0}\,\underbrace{\delta(p)}_{0} \rightsquigarrow 0.$$

If $n > 1$, then

$$\underbrace{p}_{1}\,\underbrace{\delta(g)}_{n-2} + \underbrace{g^p}_{p(n-1)}\,\delta(p) \rightsquigarrow n-1.$$

□

We might think of δ as like $\frac{\partial}{\partial p}$.

Example 2.4. (5) If $\Lambda \subseteq \mathbb{N}^n$ is a semigroup, $V[\Lambda] \subseteq V[\underline{x}]$ semigroup ring, standard LOF restricts to endomorphism of $V[\Lambda]$, so standard p-derivation restricts to $V[\Lambda]$.

(6) (Zdanowicz) If $f \in V[x_1, \dots, x_n]$ is of degree $> n$ and $k[\underline{x}]/(f)$ is normal, then $V[\underline{x}]/(f)$ does not admit a p-derivation.

Call a map $\delta : R/p^n \to R/p^n$ satisfying p-derivation axioms a p-derivation mod p^n.

Proposition 2.5 (Zdanowicz)**.** *Let $f \in V[\underline{x}]$, δ standard p-derivation on $V[\underline{x}]$. There exists p-derivation mod p^2 on $V[\underline{x}]/(f)$ if and only if $\delta(f) \in (p, f, (\frac{\partial f}{\partial x_1})^p, \dots, (\frac{\partial f}{\partial x_n})^p)$.*

Where do these show up?

Joyal: study Witt vectors.

Buivm: construction of arith jet space + intersection theory ⇒ explicit bounds on rational points.

Bhatt-Scholze: certain data of ring with p-derivations "deperfection of perfectoid ring".

Buivm-Miller: connecting jet spaces with perfectoid spaces.

Borgerm Buivm, Maniam ...

3. Jacobian Criterion

Question: How do we find the singular locus of a k-algebra?

Given:

- A ring
- $A[\underline{x}] = A[x_1, \dots, x_n]$ polynomial ring over A.
- $\underline{f} = f_1, \dots, f_m \in A[\underline{x}]$.

The Jacobian matrix

$$J(\underline{f}) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial x_n} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Then, if K is a perfect field, $R = K[\underline{x}]/I$, $I = (\underline{f})$ of pure height h, then $\mathrm{Sing}(R) = V(I_n(J(\underline{f}))R)$.

In terms of Kähler differentials, $\Omega_{R/K} = \mathrm{Coker}_R(J(\underline{f}))$, fitting ideals of $\Omega_{R/K}$ correspond to ideals of minors of $J(\underline{f})$.

$$\mathrm{Reg}(R) = \text{locus where } \Omega_{R/K} \text{ is free of "correct" rank.}$$

If we replace K by arbitrary ring A, then Jacobian criterion generalizes to detect smooth locus of $A \to R$.

Example 3.1. $R = V[x,y]/(p - xy)$, R is not smooth over V: fiber over pV is $k[x,y]/(xy)$ (not regular). But R is regular. But R is regular. $J(f) = \begin{bmatrix} -y \\ -x \end{bmatrix}$. So $V(I_1(J(f))) = V(x,y,p) = \{\mathfrak{m}\}$ contains the unique maximal ideal.

How do we find the singular locus in mixed characteristic?

Given

- A ring with p-derivation δ.
- $A[\underline{x}]$ polynomial ring over A.
- $\widetilde{\delta}$ p-derivation extending δ (any).
- $\underline{f} = f_1, \dots, f_n \in A[\underline{x}]$.

Then

$$\widetilde{J}_{\widetilde{\delta}} := \begin{bmatrix} \delta(f_1) & \cdots & \delta(f_m) \\ \left(\frac{\partial f_1}{\partial x_1}\right)^p & \cdots & \left(\frac{\partial f_m}{\partial x_1}\right)^p \\ \vdots & \ddots & \vdots \\ \left(\frac{\partial f_1}{\partial x_n}\right)^p & \cdots & \left(\frac{\partial f_m}{\partial x_n}\right)^p \end{bmatrix}$$

is the *mixed Jacobian matrix*.

Theorem 3.2 (Hochster-J)**.** *Let V be a strict p-ring (e.g., $\widehat{\mathbb{Z}}_{(p)}$), $R = V[\underline{x}]/I$, $I = (\underline{f})$ of pure height h, $\widetilde{\delta}$ p-derivation on $V[\underline{x}]$. Then,*

$$\mathrm{Sing}(R) \smallsetminus V(p) = V(I_n(J(\underline{f})R)) \smallsetminus V(p)$$
$$\mathrm{Sing}(R) \cap V(p) = V(I_n(\widetilde{J}_{\widetilde{\delta}}R)) \cap V(p)$$

Example 3.3. Let $R = V[x,y]/(p - xy)$. $\widetilde{\delta}$ standard p-derivation on $V[x,y]$.

$$\widetilde{\delta}(p - xy) = \frac{(p - x^p y^p) - (p - xy)^p}{p} = 1 + \sum_{i=1}^{p} (-1)^{i+1}\binom{p}{i}(xy)^i p^{p-i-1}$$

$$\widetilde{J}_{\widetilde{\delta}}(f) = \begin{bmatrix} -x^p \\ -y^p \end{bmatrix}$$

and $V(I_1(\widetilde{J}_{\widetilde{\delta}}(f)), p) = V(1) = \varnothing$. So R is regular.

The cokernel $\mathrm{Coker}_{R/pR}(\widetilde{J}_{\widetilde{\delta}}(\underline{f}))$ is independent of presentation of $R = V[\underline{x}]/(\underline{f})$ and choice of $\widetilde{\delta}$ p-derivation on V.

This also represents a functor with properties akin to derivations, "perivations".

Idea of using p-derivations as "missing derivations" was used earlier to characterize $P^{(n)}$ a la Zariski-Nagata [De Stefani-Grifo-J].

4. Direct summands

Let

- X_n^{gen} denote the $n \times n$ generic matrix of variables over A.
- X_n^{sym} denote the $n \times n$ symmetric matrix of variables over A.
- X_n^{alt} denote the $n \times n$ alternating matrix of variables over A.

Then

$$\begin{aligned} A[X_n^{\text{gen}}]/\det(X) &\hookrightarrow A[Y_{n\times(n-1)}, Z_{(n-1)\times n}] \\ X &\mapsto YZ \\ A[X_n^{\text{sym}}]/\det(X) &\hookrightarrow A[Y_{n\times(n-1)}] \\ X &\mapsto YY^t \\ A[X_n^{\text{alt}}]/\text{pf}(X) &\hookrightarrow A[Y_{2n\times(2n-2)}] \\ X &\mapsto Y\Omega Y^t \end{aligned}$$

If A^* is infinite, then $R = S^G$ where R is the source ring and S is the target ring in each case by some group action $G = \text{GL}_{n-1}, O_{n-1}, \text{Sp}_{2n-2}$.

These have many good properties over A where A could be a characteristic 0 field, characteristic p field or mixed characteristic DVR, e.g, they are Cohen-Macaulay (pseudorational).

Deduce from different techniques:

- standard monomial theory/ Hodge algebra [De concini, Eisenbud, Procesi,...]
- Over fields of characteristic 0, the groups G admit averaging operator, so $R \hookrightarrow S$ splits as R-modules. (direct summands of polynomial rings). [Hilbert, Noether, Hochster-Roberts,...]. In other characteristic, $R \hookrightarrow S$ do not split.

Theorem 4.1 (J-Singh)**.** *For V strict p-ring, (e.g. $\widehat{\mathbb{Z}_{(p)}}$ the classical determinantal hypersurfaces $V[X_n^{\text{gen}}]/\det$ ($n \geqslant 3$), $V[X_n^{\text{sym}}]/\det$ ($n \geqslant 3, p = 2$ or $n \geqslant 4, p \geqslant 3$) and $V[X_n^{\text{alt}}]/\text{pf}$ ($n \geqslant 3$) are not* ducks*, i.e., are not direct summands of polynomial rings by an embedding.*

Remark 4.2. Over $\mathbb{F}_p$, we don't know for any of these.

Sketch. If $R \hookrightarrow S$ are V-algebras, S has a p-derivation mod p^2 and $R \hookrightarrow S$ splits, then R has p-derivation mod p^2.

Use equational criterions to see that these classical determinantal hypersurfaces do not have p-derivations. □