LOWER BOUNDS ON HILBERT–KUNZ MULTIPLICITIES AND MAXIMAL F-SIGNATURES

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ABSTRACT. Hilbert–Kunz multiplicity and F-signature are numerical invariants of commutative rings in positive characteristic that measure severity of singularities: for a regular ring both invariants are equal to one and the converse holds under mild assumptions. A natural question is for what singular rings these invariants are closest to one. For Hilbert–Kunz multiplicity this question was first considered by the last two authors and attracted significant attention. In this paper, we study this question, i.e., an upper bound, for F-signature and revisit lower bounds on Hilbert–Kunz multiplicity.

1. INTRODUCTION

1.1. **Background.** A ring of positive characteristic has a wealth of objects arising from the Frobenius endomorphism. The focus of this paper are two numerical invariants: Hilbert–Kunz multiplicity and F-signature. For simplicity, let us assume that A is a local domain such that $A^{1/p}$ is a finitely generated A-module. The Hilbert–Kunz multiplicity of A ([22, 24]) is defined as

$$\mathrm{e}_{\mathrm{HK}}(A) := \lim_{e\to\infty} \frac{\mu_A(A^{1/p^e})}{\mathrm{rank}(A^{1/p^e})},$$

where μ_A denotes the minimal number of generators, and the F-signature of A ([18, 34]) is

$$\mathrm{s}(A) := \lim_{e\to\infty} \frac{\max\{n \mid A^{1/p^e} \cong A^{\oplus n} \oplus M\}}{\mathrm{rank}(A^{1/p^e})},$$

where M is a finitely generated A-module without free direct summands.

A fundamental result of Kunz ([21]) asserts that A^{1/p^e} is free if and only if A is regular. It follows that $\mathrm{e}_{\mathrm{HK}}(A) \geq 1$ and $1 \geq \mathrm{s}(A) \geq 0$, and under a mild condition the value is 1 if and only if A is regular ([21, 38, 18]). Furthermore, positivity of F-signature characterizes the class of strongly F-regular rings [3], a fundamental class of mild singularities that first appeared in the tight closure theory [17]. A related result of Blickle–Enescu [4] shows that small Hilbert–Kunz multiplicity also forces the ring to be strongly F-regular.

A natural question is how close can the Hilbert–Kunz multiplicity of a singularity be to 1? And a natural guess is that the simplest double point singularity $k[[x_1,\dots,x_d]]/(x_1^2+\cdots+x_d^2)$ should have the smallest Hilbert–Kunz multiplicity (see, Conjecture 2.4 for details). By [39, 41, 2] this is now a theorem in dimension at most 6.

In this paper, we extend this investigation by asking to find further bounds on Hilbert–Kunz multiplicity of mild singularities and considering the analogous question for F-signature. For instance, in dimension 2, most of non-regular F-regular local rings are quotient singularities, in which case we have that $\mathrm{s}(A) = 1/|G| \leq 1/2$, where $A = k[[x,y]]^G$ and G is a finite subgroup of $\mathrm{GL}_2(k)$. It seems that a similar question has no answer even in dimension 3.

Question 1.1. Let A be an F-regular local domain of dimension $d \geq 3$ which is not regular. Then what is the upper bound on $\mathrm{s}(A)$?

We give a partial answer to the question above, and pose a conjecture; see Conjecture 2.10. Let us explain the organization of the paper.

1.2. **Structure of the paper and main results.** In Section 2, we recall several definitions (Hilbert–Kunz multiplicity, F-regularity, FFRT, F-signature and so on) and pose two conjectures. In Section 3, we give a lower bound on Hilbert–Kunz multiplicities. Namely, we prove the following theorem and its refinement for 3-dimensional case (see Theorem 3.7).

Theorem 1.2 (= 3.2). *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of characteristic $p>0$. If $d=\dim A\geq 3$, then we have*

$$\mathrm{e}_{\mathrm{HK}}(I) > \frac{\mathrm{e}(I)+d}{d!}.$$

In Section 4, we prove that $\mathrm{s}(A)\leq\frac{1}{2}$ for non-Gorenstein Cohen-Macaulay local rings A (see Proposition 4.1) and characterize the extreme case.

Theorem 1.3 (= 4.6). *Let A be a Cohen-Macaulay local domain but not Gorenstein. Then $\mathrm{s}(A)\leq\frac{1}{2}$, and the following conditions are equivalent*:

(1) $\mathrm{s}(A)=\frac{1}{2}$.
(2) *F^e_*A is a finite direct sum of A and ω_A for every $e\geq 1$.*

When this is the case, $\mathrm{e}_{\mathrm{HK}}(A)=\frac{\mathrm{type}(A)+1}{2}$. Moreover, if, in addition, either A is $\mathbb{Q}$-Gorenstein or a toric singularity, then it is isomorphic to the Veronese subring $k[[x_1,x_2,\ldots,x_d]]^{(2)}$, where $k[[x_1,\ldots,x_d]]^{(n)}=k[[(x_1,\ldots,x_d)^n]]$ (see Theorem 4.17 and 5.6).

The F-signature of a Gorenstein ring may exceed $\frac{1}{2}$. We explore an upper bound on F-signature for Gorenstein, non-regular local rings of dimension three.

Theorem 1.4 (= 4.18). *Let $(A,\mathfrak{m},k)$ be a 3-dimensional Gorenstein F-regular local ring with $\mathrm{e}(A)\geq 3$. Then $\mathrm{s}(A)\leq\frac{\mathrm{e}(A)}{24}$.*

We are able to determine the maximal F-signature of 3-dimensional toric singularity.

Theorem 1.5 (= 5.9). *Let A be a 3-dimensional non-regular Gorenstein toric ring. The following conditions are equivalent;*

(1) $\mathrm{s}(A)>\frac{1}{2}$,
(2) *A is isomorphic to $k[x,y,z,w]/(xy-zw)$.*

When this is the case, $\mathrm{s}(A)=\frac{2}{3}$.

2. PRELIMINARIES

Let $(A,\mathfrak{m})$ be a local ring of characteristic $p>0$ and let $F^e\colon A\to A$ denote the e^{th} iterated Frobenius map of A. For an A-module M, the Frobenius push-forward of M, $F^e_*M=\{F^e_*m\mid m\in M\}$, is defined as follows: it agrees with M as an abelian group and A acts by $a\cdot F^e_*m=F^e_*(a^{p^e}m)$ for any $a\in A$ and $m\in M$. If A is reduced, F^e_*A is identified with A^{1/p^e} which consists of p^e-th roots of A. The ring A is called *F-finite* if F^e_*A is a finitely generated A-module for every (some) $e\geq 1$.

We now recall a more general definition of Hilbert–Kunz multiplicity.

Definition 2.1. Let $\ell_A(W)$ denote the length of a finitely generated A-module W. For an $\mathfrak{m}$-primary ideal $I\subset A$ we denote $I^{[q]}=(a^q\mid a\in I)A$ for each $q=p^e$. If M is a finitely generated A-module,

$$\mathrm{e}(I,M):=\lim_{n\to\infty}\frac{d!}{n^d}\ell_A(M/I^{n+1}M)\quad(\text{resp. }\mathrm{e}_{\mathrm{HK}}(I,M):=\lim_{q\to\infty}\frac{\ell_A(M/I^{[q]}M)}{q^d})$$

is called the *multiplicity* (resp. the *Hilbert–Kunz multiplicity*) of M with respect to I. For brevity, we denote $\mathrm{e}(I)=\mathrm{e}(I,A)$ (resp. $\mathrm{e}_{\mathrm{HK}}(I)=\mathrm{e}_{\mathrm{HK}}(I,A)$) and call it the *multiplicity* (resp. the *Hilbert–Kunz multiplicity*) of I. We also denote, $\mathrm{e}(\mathfrak{m},M)=\mathrm{e}(M)$ and $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m},M)=\mathrm{e}_{\mathrm{HK}}(M)$.

Recall the fundamental properties of Hilbert–Kunz multiplicities; see e.g. [38].

Proposition 2.2 ([38, (2.3),(2.4),(2.5)], [12]). *Let $I \subset A$ be an $\mathfrak{m}$-primary ideal.*

(1) *The following inequalities hold true*:
$$\frac{\mathrm{e}(I)}{d!} \leq \mathrm{e}_{\mathrm{HK}}(I) \leq \mathrm{e}(I).$$
If, in addition, $d \geq 3$, then $\frac{\mathrm{e}(I)}{d!} < \mathrm{e}_{\mathrm{HK}}(I)$.

(2) *If I is a parameter ideal, then $\mathrm{e}_{\mathrm{HK}}(I) = \mathrm{e}(I)$.*

(3) *Let $\mathrm{Assh}(A)$ denote the set of all associated prime ideals P with $\dim A/P = \dim A$. Then*
$$\mathrm{e}_{\mathrm{HK}}(I,M) = \sum_{P \in \mathrm{Assh}(A)} \mathrm{e}_{\mathrm{HK}}(I,A/P) \cdot \ell_{A_P}(M_P).$$

2.1. **Minimal value of Hilbert–Kunz multiplicity.** Now we want to discuss the conjectural lower bound on Hilbert–Kunz multiplicities of singularities. In order to state it, we recall the definition of type (A_1) simple singularity.

Definition 2.3. Let p be a prime number, k be an algebraically closed field of characteristic p, and d a positive integer. Then we define $A_{p,d}$ as follows:
$$A_{p,d} := \begin{cases} k[[x_0,x_1,\dots,x_d]]/(x_0x_1 + x_2x_3 + \cdots + x_{d-1}x_d) & \text{(when } d = 2m-1,\ m \geq 1); \\ k[[x_0,x_1,\dots,x_d]]/(x_0^2 + x_1x_2 + x_3x_4 + \cdots + x_{d-1}x_d) & \text{(when } d = 2m,\ m \geq 1). \end{cases}$$

For $p > 2$ the equation takes a more familiar form $A_{p,d} \cong k[[x_0,x_1,\dots,x_d]]/(x_0^2 + x_1^2 + \cdots + x_d^2)$. Han and Monsky ([11]) gave an algorithm computing $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ for given $p > 2, d$. However, a closed formula for $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ is only known for small values of d. However, Gessel and Monsky ([10]) showed that $\lim_{p\to\infty} \mathrm{e}_{\mathrm{HK}}(A_{p,d}) = 1 + c_d$ where
$$\sec x + \tan x = 1 + \sum_{i=1}^{\infty} c_d x^d \quad \left(|x| < \frac{\pi}{2}\right).$$

The first several values of c_d are recorded in Table 1 below.

Conjecture 2.4 (cf. [41, Conjecture 4.2]). Let $(A,\mathfrak{m},k)$ be an F-finite, formally unmixed, non-regular local ring of dimension $d \geq 1$. Then

(1) $\mathrm{e}_{\mathrm{HK}}(A) \geq \mathrm{e}_{\mathrm{HK}}(A_{p,d}) \geq 1 + c_d$, where c_d is defined above.

(2) Suppose that $k = \overline{k}$. If $\mathrm{e}_{\mathrm{HK}}(A) = \mathrm{e}_{\mathrm{HK}}(A_{p,d})$, then $\widehat{A} \cong A_{p,d}$.

Let us summarize the cases where Conjecture 2.4 is known.

Theorem 2.5. *Let A be a formally unmixed, non-regular local ring and p is an odd prime number.*

(1) *If $d \leq 3$ then Conjecture 2.4 holds and* $\mathrm{e}_{\mathrm{HK}}(A_{p,d}) = 1 + c_d$ ([39, Theorem 3.1], [41, Theorem 3.1]). *In fact, these results also show that* $\mathrm{e}_{\mathrm{HK}}(A) \geq \mathrm{e}_{\mathrm{HK}}(A_{p,d}) = 1 + c_d$ *for $p = 2$.*

(2) *If $d = 4$, then Conjecture 2.4 holds* ([41, Theorem 4.3]) *but* $\mathrm{e}_{\mathrm{HK}}(A_{p,4}) = \frac{29p^2+15}{24p^2+12} > \frac{29}{24}$ *now depends on p* ([10]).

(3) *If $d = 5,6$ then* $\mathrm{e}_{\mathrm{HK}}(A) \geq \mathrm{e}_{\mathrm{HK}}(A_{p,d}) \geq 1 + c_d$ ([2, Theorem 5.2]).

(4) *If A is a complete intersection local ring, then* $\mathrm{e}_{\mathrm{HK}}(A) \geq \mathrm{e}_{\mathrm{HK}}(A_{p,d})$ (*see* [8, Theorem 4.6]).

(5) *Yoshida* ([44]) *conjectures that $\mathrm{e}_{\mathrm{HK}}(A_{p,d})$ is a decreasing function in p for a fixed d, thus the second part of the conjecture would imply the first.*

Observation 2.6. If $p=2$ and $d=2m$ ($m=1,2,\ldots$), then the following statement can be proved by using an argument in [11] (see [44] for details)

$$\ell\left(A_{2,d}/\mathfrak{m}^{[2^e]}\right)=\frac{2^m+1}{2^m}\,2^{de}.$$

In particular, $e_{HK}(A_{2,d})=\frac{2^m+1}{2^m}$.

Similarly, [44] conjectures that if $d=2m-1$ ($m=1,2,\cdots$), then

$$\ell(A_{2,d}/\mathfrak{m}^{[2^e]})=\frac{2^m}{2^m-1}\,2^{de}-\frac{(2^{m-1})^e}{2^m-1}$$

for every $e\geq 1$. In particular, if would follow that $e_{HK}(A_{2,d})=\frac{2^m}{2^m-1}$.

Based upon these observations, we pose an improved conjecture as follows:

Conjecture 2.7. Let $(A,\mathfrak{m},k)$ be a formally unmixed non-regular local ring of dimension $d\geq 1$ and with algebraically closed residue field. Let $m\geq 1$ be an integer.

(1) If $d=2m-1$, then either $\widehat{A}\cong A_{p,d}$ or $e_{HK}(A)>\frac{2^m}{2^m-1}$.
(2) If $d=2m$, then either $\widehat{A}\cong A_{p,d}$ or $e_{HK}(A)>\frac{2^m+1}{2^m}$.

By results of Watanabe and Yoshida, Conjecture 2.7 has an affirmative answer when $p\geq 3$ and $d\leq 4$. The following table depicts the difference between two conjectures.

d	1	2	3	4	5	6
$1+c_d$	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{29}{24}$	$\frac{17}{15}$	$\frac{781}{720}$
(RHS)	2	$\frac{3}{2}$	$\frac{4}{3}$	$\frac{5}{4}$	$\frac{8}{7}$	$\frac{9}{8}$

TABLE 1. Comparison between the two conjectured bounds.

2.2. **Strong F-regularity and F-signature.** Hilbert–Kunz multiplicity is inherently connected with tight closure, a powerful theory developed by Hochster and Huneke in a series of papers starting at [16].

Definition 2.8 (cf. [16])**.** Let $I\subset A$ be an ideal, and let x be an element of A. Put $A^o=A\setminus\cup_{P\in\mathrm{Min}(A)}P$. For $x\in A$, we say that x is in the *tight closure* of I (denoted by I^*) if there exists an element $c\in A^o$ such that $cx^q\in I^{[q]}$ for sufficiently large $q=p^e$.

A local ring A is said to be weakly F-regular (resp. F-rational) if any ideal I (resp. any parameter ideal I) is tightly closed, that is, $I^*=I$.

A result of Hochster and Huneke [16, Theorem 8.17] asserts that $e_{HK}(I^*)=e_{HK}(I)$ and, moreover, I^* is the largest ideal containing I with same Hilbert–Kunz multiplicity.

On the other hand, F-signature coincides with the *minimal relative Hilbert–Kunz multiplicity* [40, 43, 26]. and is connected to the following class of singularities.

Definition 2.9 (cf. [17])**.** A local ring A is called *strongly F-regular* if for any $c\in A^o$, there exists $q=p^e$, $e\geq 1$ such that the map $A\hookrightarrow A^{1/q}$ defined by $x\mapsto c^{1/q}x$ splits as an A-linear map. Any Noetherian ring A is called *strongly F-regular* if any localization of A is also a (strongly) F-regular local ring.

Strongly F-regular singularities enjoy many nice properties and are always normal and Cohen-Macaulay. For example, quotient singularities and toric singularities are strongly F-regular rings. As it was already mentioned, $s(A)>0$ if and only if A is strongly F-regular by a result of Aberbach and Leuschke [3, Theorem 0.2].

The simple singularity $A_{p,d}$ discussed above is a hypersurface with $\mathrm{e}(A_{p,d}) = 2$, thus by [40, Example 2.3] $\mathrm{e_{HK}}(A) = 2 - \mathrm{s}(A)$ and $\mathrm{s}(A)$ attends the maximal value if and only if $\mathrm{e_{HK}}(A)$ is minimal. The following conjecture is then natural.

Conjecture 2.10. Let $(A, \mathfrak{m})$ be a non-regular local ring of dimension $d \geq 1$. Then

$$\mathrm{s}(A) \leq 2 - \mathrm{e_{HK}}(A_{p,d}) = \mathrm{s}(A_{p,d}).$$

The theory of F-signature originates in the following particular case of rings of finite F-representation type, which was introduced by Smith and Van den Bergh [32] (see also [42]).

Definition 2.11. We say that A has *finite F-representation type* ($FFRT$) if there is a finite set $\mathscr{S} = \{M_0, M_1, \ldots, M_n\}$ of isomorphism classes of indecomposable finitely generated A-modules such that for any positive integer e, $F^e_* A$ is isomorphic to a finite direct sum of these modules, that is,

$$F^e_* A \cong M_0^{\oplus c_{0,e}} \oplus M_1^{\oplus c_{1,e}} \oplus \cdots \oplus M_n^{\oplus c_{n,e}}$$

for some $c_{i,e} \in \mathbb{Z}_{\geq 0}$. Moreover, we say that a finite set $\mathscr{S} = \{M_0, M_1, \ldots, M_n\}$ is the ($FFRT$) *system* of A if every A-module M_i appears non-trivially in $F^e_* A$ as a direct summand for some $e \in \mathbb{N}$.

3. Lower bound on Hilbert–Kunz multiplicities

The last two authors gave a lower bound on Hilbert–Kunz multiplicities of two-dimensional unmixed (Cohen-Macaulay) local rings A in terms of usual multiplicities:

$$\mathrm{e_{HK}}(I) \geq \frac{\mathrm{e}(I) + 1}{2}$$

for any $\mathfrak{m}$-primary ideal I of A [39]. In this section, we consider a higher dimensional analogue of this inequality; see Theorem 3.2.

We recall [2, Theorem 3.2] which improves the volume estimation technique developed in [41]. For any real number s we define $v_{s,d}$ to be the volume of $\{(x_1, \ldots, x_d) \in [0,1]^d \mid \sum_{i=1}^d x_i \leq s\}$ which can be computed as

$$v_{s,d} = \sum_{n=0}^{\lfloor s \rfloor} (-1)^n \frac{(s-n)^d}{(d-n)!\, n!},$$

where $\lfloor \quad \rfloor$ stands for round down.

For an element $x \in A$ we denote

$$\overline{v}_I(x) := \lim_{n \to \infty} \frac{\sup\{k \mid x^n \in I^k\}}{n}.$$

It is known that the limit exists and $\overline{v}_I(x) \geq k$ if and only if $x \in \overline{I^k}$; see Rees [28].

Theorem 3.1 (Aberbach–Enescu[2])**.** *Let $(A, \mathfrak{m})$ be a formally unmixed reduced local ring of characteristic $p > 0$ and dimension d. Let J be a minimal reduction of an $\mathfrak{m}$-primary ideal I and let r be an integer such that $r \geq \mu_A(I/J^*)$. For every real number $s \geq 0$, we have*

$$\mathrm{e_{HK}}(I) \geq \mathrm{e}(I) \left(v_{s,d} - \sum_{i=1}^{r} v_{s-t_i,d} \right),$$

where $t_i = \overline{v}_I(z_i)$ for $z_1, \ldots, z_r$ generators of I modulo J^*.

In particular,

$$\mathrm{e_{HK}}(I) \geq \mathrm{e}(I) \left(v_{s,d} - r \cdot v_{s-1,d} \right).$$

Using the above theorem and the technique developed in [1], we can improve Proposition 2.2.

Theorem 3.2. *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of characteristic $p>0$. If $d=\dim A\geq 3$, then we have*

$$\mathrm{e}_{\mathrm{HK}}(I)>\frac{\mathrm{e}(I)+d}{d!}.$$

In what follows, we may assume that A is complete and the residue field $k=A/\mathfrak{m}$ is infinite. Let I denote an $\mathfrak{m}$-primary ideal and J its minimal reduction.

Lemma 3.3. *Suppose that A is Cohen-Macaulay and $I=I^*$, that is, I is tightly closed. If $\mathfrak{m}I\not\subset J$, then $\mu_A(I/J^*)\leq \mathrm{e}(I)-2$.*

Proof. By definition, $\mu_A(I/J^*)=\ell_A(I/J^*+\mathfrak{m}I)\leq \ell_A(I/J+\mathfrak{m}I)$. Thus by assumption, we have

$$\mu_A(I/J^*)\leq \ell_A(I/J+\mathfrak{m}I)\leq \ell_A(I/J)-1=\ell_A(A/J)-\ell_A(A/I)-1\leq \mathrm{e}(I)-2,$$

where we can use $\mathrm{e}(I)=\ell_A(A/J)$ since A is Cohen-Macaulay. □

The following proposition gives a refinement of Aberbach and Enescu [1, Corollary 3.4].

Proposition 3.4. *Let $(A,\mathfrak{m})$ be a Cohen-Macaulay local ring of dimension $d\geq 1$. Let I be an $\mathfrak{m}$-primary ideal and suppose that there exists a minimal reduction J of I such that $I^2\subset J$ (e.g. I is stable). Then*

$$\mathrm{e}_{\mathrm{HK}}(I)\geq \frac{\mathrm{e}(I)}{2}.$$

Proof. Let ω_A denote the canonical module of A. Since $I^{[q]}\omega_A\subseteq J^{[q]}\omega_A : I^{[q]}$ for any $q=p^e$ by assumption, we get

$$\begin{aligned}\ell_A(\omega_A/J^{[q]}\omega_A) &= \ell_A(\omega_A/J^{[q]}\omega_A : I^{[q]})+\ell_A(J^{[q]}\omega_A : I^{[q]}/J^{[q]}\omega_A)\\ &\leq \ell_A(\omega_A/I^{[q]}\omega_A)+\ell_A(J^{[q]}\omega_A : I^{[q]}/J^{[q]}\omega_A).\end{aligned}$$

Then

$$\lim_{q\to\infty}\frac{\ell_A(\omega_A/J^{[q]}\omega_A)}{q^d}=\mathrm{e}_{\mathrm{HK}}(J;\omega_A)=\mathrm{e}_{\mathrm{HK}}(J)=\mathrm{e}(I),\qquad \lim_{q\to\infty}\frac{\ell_A(\omega_A/I^{[q]}\omega_A)}{q^d}=\mathrm{e}_{\mathrm{HK}}(I;\omega_A)=\mathrm{e}_{\mathrm{HK}}(I).$$

On the other hand, since

$$(J^{[q]}\omega_A : I^{[q]})/J^{[q]}\omega_A\cong \mathrm{Hom}_{A/J^{[q]}}(A/I^{[q]},\omega_A/J^{[q]}\omega_A)\cong \mathrm{Hom}_{A/J^{[q]}}(A/I^{[q]},\omega_{A/J^{[q]}})\cong \omega_{A/I^{[q]}},$$

we get

$$\ell_A((J^{[q]}\omega_A : I^{[q]})/J^{[q]}\omega_A)=\ell_A(\omega_{A/I^{[q]}})=\ell_A(A/I^{[q]})$$

by Matlis duality. Hence

$$\lim_{q\to\infty}\frac{\ell_A((J^{[q]}\omega_A : I^{[q]})/J^{[q]}\omega_A)}{q^d}=\mathrm{e}_{\mathrm{HK}}(I)$$

and thus $\mathrm{e}(I)\leq 2\cdot \mathrm{e}_{\mathrm{HK}}(I)$, as required. □

Definition 3.5. A Cohen-Macaulay local ring $(A,\mathfrak{m})$ is said to have *minimal multiplicity* if $\mu_A(\mathfrak{m})=\mathrm{e}(A)+\dim A-1$. This condition is equivalent that $\mathfrak{m}$ is stable, that is, $\mathfrak{m}^2=J\mathfrak{m}$ for some minimal reduction J of $\mathfrak{m}$.

Corollary 3.6 ([1])**.** *Let A be a Cohen-Macaulay local ring with minimal multiplicity, that is, $\mu_A(\mathfrak{m})=\mathrm{e}(A)+\dim A-1$. Then $\mathrm{e}_{\mathrm{HK}}(A)\geq \frac{\mathrm{e}(A)}{2}$.*

Proof of Theorem 3.2. We may assume that I is tightly closed because $\mathrm{e}_{\mathrm{HK}}(I)=\mathrm{e}_{\mathrm{HK}}(I^*)$ and $\mathrm{e}(I)=\mathrm{e}(I^*)$. Moreover, we may assume $\mathrm{e}(I)\geq 2$.

Case 1. The case of $I^2\subset J$.

We can apply Proposition 3.4 to obtain $e_{HK}(I) \geq \dfrac{e(I)}{2} > \dfrac{e(I)+d}{d!}$ if $d \geq 3$ and $e(I) \geq 2$.

Case 2. The case of $I^2 \not\subset J$.

By Lemma 3.3, we have $\mu_A(I/J^*) \leq e(I) - 2$. So we can apply Theorem 3.1 as $e = e(I) \geq 2$, $r = e-2$ and $s = 1 + \dfrac{1}{e}$. Then $\lfloor s \rfloor = 1$ and

$$\begin{aligned} d! \cdot e^d (v_s - r \cdot v_{s-1}) &= e^d \cdot d! \cdot \left(\frac{(1+1/e)^d}{d!} - \frac{(1/e)^d}{(d-1)!} - (e-2)\frac{(1/e)^d}{d!} \right) \\ &= (e+1)^d - d - e + 2 \\ &= e^d + de^{d-1} + \sum_{k=2}^{d-2} \binom{d}{k} e^k + de + 1 - d - e + 2 \\ &\geq e^d + de^{d-1} + (d-1)(e-1) + 2 > e^{d-1}(e+d). \end{aligned}$$

Hence $e_{HK}(I) \geq e(v_s - r \cdot v_{s-1}) > \dfrac{e+d}{d!}$, as required. ☐

If we fix d, then this is *not* the best possible. In this paper, we prove the following theorem, which gives the optimal bound on the Hilbert-Kunz multiplicity $e_{HK}(A)$ in dimension 3.

Theorem 3.7. *Let $(A, \mathfrak{m}, k)$ be an unmixed local ring of dimension 3 and characteristic $p > 0$. Then*

$$e_{HK}(A) \geq \frac{e(A)}{6} + 1.$$

If equality holds, then A is a strongly F-regular local ring with $e(A) = 2$. Moreover, if, in addition, the residue field k is algebraically closed and $p \geq 3$, then $\widehat{A} \cong k[[x,y,z,w]]/(xz - yw)$ and $e_{HK}(A) = \frac{4}{3}$.

In order to prove the theorem, we prove a stronger result as follows.

Lemma 3.8. *Under the assumptions of Theorem 3.7, we suppose that $e = e(A) \geq 3$. Then*

(1) $e_{HK}(A) \geq \dfrac{e}{6} \left(\dfrac{e+2+\sqrt{e+2}}{e+1} \right)^2$.

(2) *If A is neither F-rational nor Cohen-Macaulay with minimal multiplicity, then*

$$e_{HK}(A) \geq \frac{1}{6} \left(e + 3 + \frac{2}{e} + \left(2 + \frac{2}{e} \right) \sqrt{e+1} \right).$$

Proof. (1) For $1 \leq s \leq 2$, we will optimize the volume estimate

$$\begin{aligned} e_{HK}(A) &\geq e \cdot (v_s - (e-1)v_{s-1}) \\ &= e \cdot \left(\frac{s^3}{6} - \frac{(s-1)^3}{2} - (e-1)\frac{(s-1)^3}{6} \right) \\ &= \frac{e(s^3 - (e+2)(s-1)^3)}{6}. \end{aligned}$$

We consider a function $f(s) = s^3 - (e+2)(s-1)^3$. The derivative is given by $f'(s) = 3s^2 - 3(e+2)(s-1)^2$ and the equation $f'(s) = 0$ has roots $s_{\pm} = \frac{e+2 \pm \sqrt{e+2}}{e+1}$. Since $s_- < 1 < s_+ < 2$, the maximum on $1 \leq s \leq 2$ is at s_+ which gives the inequality:

$$e_{HK}(A) \geq \frac{e}{6} \cdot f(s_+) = \frac{e}{6} \cdot s_+^2 = \frac{e}{6} \left(\frac{e+2+\sqrt{e+2}}{e+1} \right)^2$$

(2) Under the assumption, we have $\mathrm{e}_{\mathrm{HK}}(A) \geq \mathrm{e} \cdot (v_s - (\mathrm{e}-2)v_{s-1})$ and $\mathrm{e} \geq 3$. So if we consider $g(s) = s^3 - (\mathrm{e}+1)(s-1)^3$, then $1 \leq \frac{\mathrm{e}+1+\sqrt{\mathrm{e}+1}}{\mathrm{e}} \leq 2$ and a similar argument as above implies

$$\mathrm{e}_{\mathrm{HK}}(A) \geq \frac{\mathrm{e}}{6} \cdot g\left(\tfrac{\mathrm{e}+1+\sqrt{\mathrm{e}+1}}{\mathrm{e}}\right) = \frac{1}{6}\left(\mathrm{e}+3+\frac{2}{\mathrm{e}}+\left(2+\frac{2}{\mathrm{e}}\right)\sqrt{\mathrm{e}+1}\right),$$

as required. □

Proof of Theorem 3.7. First suppose that A is neither F-rational nor Cohen-Macaulay with minimal multiplicity. If $\mathrm{e}=2$, then $\mathrm{e}_{\mathrm{HK}}(A) = 2 > \frac{4}{3} = \frac{\mathrm{e}}{6}+1$. Hence we may assume $\mathrm{e} = \mathrm{e}(A) \geq 3$. Then Lemma 3.8 yields that

$$\mathrm{e}_{\mathrm{HK}}(A) \geq \frac{1}{6}\left(\mathrm{e}+3+\frac{2}{\mathrm{e}}+\left(2+\frac{2}{\mathrm{e}}\right)\sqrt{\mathrm{e}+1}\right) > \frac{\mathrm{e}}{6}+1.$$

Next suppose that A is F-rational and Cohen-Macaulay with minimal multiplicity.
If $\mathrm{e} \geq 4$, then $\mathrm{e}_{\mathrm{HK}}(A) \geq \frac{\mathrm{e}}{2} > \frac{\mathrm{e}}{6}+1$.
If $\mathrm{e}=3$, then [41, Lemma 3.3(3)] implies $\mathrm{e}_{\mathrm{HK}}(A) \geq \frac{13}{8} > \frac{3}{2} = \frac{\mathrm{e}}{6}+1$.
Suppose that $\mathrm{e}=2$. Then the main theorem in [41] yields $\mathrm{e}_{\mathrm{HK}}(A) \geq \frac{4}{3} = \frac{\mathrm{e}}{6}+1$ and equality holds if and only if $\widehat{A} \cong k[[x,y,z,w]]/(xz-yw)$. Therefore we complete the proof. □

4. Upper bounds on F-signature

The main aim of this section is to give an upper bound on F-signature for non-Gorenstein Cohen-Macaulay local rings. We start with a few preliminaries.

Let M be an MCM A-module. Then $\mu_A(M) \leq \mathrm{e}(M)$, because multiplicity can be computed from a regular sequence. We say that M is an *Ulrich A-module* if $\mu_A(M) = \mathrm{e}(M)$. Ulrich modules first appeared in [5] under the name *maximally generated maximal Cohen-Macaulay module*.

If A is a local ring of positive characteristic $p>0$ and M is a finitely generated A-module then the rank of the largest free summand of M is independent of a decomposition, because we may pass to the completion, see [25, Remark 3.4]. Moreover, if A is a Cohen-Macaulay local ring with the canonical module ω_A and M is maximal Cohen-Macaulay, then the number of direct summand of M isomorphic to ω_A is also independent of a direct decomposition, since these correspond to a free summand of $\mathrm{Hom}_A(M, \omega_A)$. Last, we note that an F-finite Cohen-Macaulay ring has a canonical module by a result of Gabber [9, Remark 13.6].

The second assertion of the next proposition was initially observed by De Stefani and Jeffries in relation with Sannai's dual F-signature ([30]).

Proposition 4.1 (cf. [23])**.** *Let A be an F-finite Cohen-Macaulay local domain which is not Gorenstein. Then*

$$\mathrm{e}_{\mathrm{HK}}(A) \leq \mathrm{s}(A)(\mathrm{type}(A)+1) + 2\cdot \mathrm{e}(A)\left(\frac{1}{2}-\mathrm{s}(A)\right).$$

In particular, $\mathrm{s}(A) \leq \dfrac{1}{2}$.

Proof. For every $e \geq 1$, we can write

$$F_*^e A = A^{\oplus a_e} \oplus \omega_A^{\oplus b_e} \oplus M_e,$$

where a_e, b_e are non-negative integers and M_e is a maximal Cohen-Macaulay A-module that does not contain A and ω_A as direct summands. Then

$$F_*^e \omega_A \cong \mathrm{Hom}_A(F_*^e A, \omega_A) \cong A^{\oplus b_e} \oplus \omega_A^{\oplus a_e} \oplus \mathrm{Hom}_A(M_e, \omega_A).$$

By the argument in the proof of Sannai [30, Proposition 3.10], we have

$$\lim_{e\to\infty} \frac{a_e}{\operatorname{rank} F^e_* A} = \lim_{e\to\infty} \frac{b_e}{\operatorname{rank} F^e_* A} = \mathrm{s}(A).$$

Note that $\mu_A(F^e_* A) = p^{e\alpha(A)} \cdot \ell_A(A/\mathfrak{m}^{[p^e]})$ and $\operatorname{rank}_A F^e_* A = p^{e(\alpha(A)+d)}$.

Since M_e is a maximal Cohen-Macaulay A-module, we have

$$\begin{aligned} \mu_A(F^e_* A) &= a_e + b_e \cdot \operatorname{type}(A) + \mu_A(M_e) \\ &\leq a_e + b_e \cdot \operatorname{type}(A) + \mathrm{e}_A(M_e) \\ &= a_e + b_e \cdot \operatorname{type}(A) + \mathrm{e}(A) \operatorname{rank}_A(M_e) \\ &= a_e + b_e \cdot \operatorname{type}(A) + \mathrm{e}(A)(\operatorname{rank} F^e_* A - a_e - b_e). \end{aligned} \tag{4.1}$$

Hence

$$\frac{\ell_A(A/\mathfrak{m}^{[p^e]})}{p^{ed}} = \frac{\mu_A(F^e_* A)}{\operatorname{rank} F^e_* A} \leq \frac{a_e}{\operatorname{rank} F^e_* A} + \operatorname{type}(A) \cdot \frac{b_e}{\operatorname{rank} F^e_* A} + \mathrm{e}(A) \left(1 - \frac{a_e}{\operatorname{rank} F^e_* A} - \frac{b_e}{\operatorname{rank} F^e_* A}\right),$$

and the first assertion follows after taking limits as e tends to ∞.

In particular, since

$$0 \leq \frac{\operatorname{rank}_A(M_e)}{\operatorname{rank} F^e_* A} = 1 - \frac{a_e}{\operatorname{rank} F^e_* A} - \frac{b_e}{\operatorname{rank} F^e_* A},$$

we get $0 \leq 1 - 2 \cdot \mathrm{s}(A)$, that is, $\mathrm{s}(A) \leq \frac{1}{2}$. □

Remark 4.2. We note that there are classes of Gorenstein rings having F-signature less than or equal to $\frac{1}{2}$. For example, if A is a 2-dimensional Gorenstein strongly F-regular local ring, then A is a hypersurface and has minimal multiplicity, thus $\mathrm{e}(A) = 2$. Therefore, we have $\mathrm{s}(A) = 2 - \mathrm{e}_{\mathrm{HK}}(A) \leq 2 - \frac{3}{2} = \frac{1}{2}$ by [41, Corollary 2.6] (see also [39, Example 4.1] and [18, Example 18]).

The following theorem characterizes the equality in Proposition 4.1 in terms of the FFRT property.

Theorem 4.3. *Let A be a strongly F-regular local domain of type at least two (i.e., not Gorenstein). The following conditions are equivalent*:

(1) $\mathrm{e}_{\mathrm{HK}}(A) = \mathrm{s}(A)(\operatorname{type}(A)+1) + 2 \cdot \mathrm{e}(A)\left(\frac{1}{2} - \mathrm{s}(A)\right)$.
(2) *$F^e_* A$ is a finite direct sum of A, ω_A and an Ulrich A-module for every $e \geq 1$.*

Proof. For every $e \geq 1$, we can write $F^e_* A = A^{\oplus a_e} \oplus \omega_A^{\oplus b_e} \oplus M_e$, where a_e and b_e are non-negative integers and M_e is a maximal Cohen-Macaulay A-module that does not contain A and ω_A as direct summands.

(2) $\Longrightarrow$ (1): By the assumption, M_e is an Ulrich A-module, that is, $\mu_A(M_e) = \mathrm{e}(M_e)$. Hence the assertion follows from the proof of Proposition 4.1.

(1) $\Longrightarrow$ (2): Suppose that there exists e' such that $F^{e'}_* A = A^{\oplus a_{e'}} \oplus \omega_A^{\oplus b_{e'}} \oplus M_{e'}$, where $M_{e'}$ is an MCM A-module but not an Ulrich A-module, namely, $\mu_A(M_{e'}) < \mathrm{e}(M_{e'})$. By [25, Lemma 3.3] we may now build a similar decomposition for $e \geq e'$:

$$F^e_* A = A^{\oplus a_e} \oplus \omega^{\oplus b_e} \oplus M_{e'}^{\oplus c_e} \oplus N_e$$

such that $\liminf_{e\to\infty} c_e / \operatorname{rank} F^e_* A > 0$. Following the proof of Proposition 4.1 we obtain that

$$\mu_A(F^e_* A) \leq a_e + b_e \cdot \operatorname{type}(A) + c_e(\mathrm{e}(M_e) - \mu_A(M_e)) + \mathrm{e}(A)(\operatorname{rank} F^e_* A - a_e - b_e),$$

which shows after dividing by $\operatorname{rank} F^e_* A$ and passing to the limit that

$$\mathrm{e}_{\mathrm{HK}}(A) < \mathrm{s}(A)(\operatorname{type}(A)+1) + \mathrm{e}(A)(1 - 2 \cdot \mathrm{s}(A)).$$

□

One can prove the following proposition by a similar method as in the proof of Proposition 4.1 and Theorem 4.3.

Proposition 4.4. *Suppose that A is a Gorenstein local ring of dimension $d \geq 2$. Then*

$$\mathrm{e_{HK}}(A) \leq \mathrm{s}(A) + (1 - \mathrm{s}(A)) \cdot \mathrm{e}(A)$$

and the equality holds if and only if $F^e_* A$ can be written as a direct sum of A and Ulrich A-modules for every $e \geq 1$.

We note that if $\mathrm{e}(A) = 2$ we have $\mathrm{e_{HK}}(A) = 2 - \mathrm{s}(A)$, and hence A satisfies the equality of Proposition 4.4.

Question 4.5. If A satisfies $\mathrm{e_{HK}}(A) = \mathrm{s}(A) + (1 - \mathrm{s}(A)) \cdot \mathrm{e}(A)$, then is $\mathrm{e}(A) \leq 2$?

We proceed to study non-Gorenstein rings whose F-signature is $\frac{1}{2}$.

Theorem 4.6. *Let A be a Cohen-Macaulay local domain but not Gorenstein. Then the following conditions are equivalent*:

(1) $\mathrm{s}(A) = \frac{1}{2}$,
(2) *A is FFRT with the FFRT system $\{A, \omega_A\}$.*

When this is the case, $\mathrm{e_{HK}}(A) = \frac{\mathrm{type}(A)+1}{2}$.

Proof. (2) $\Longrightarrow$ (1) essentially follows from the proof of Proposition 4.1, because in this case there is no M_e and we have equality throughout.

(1) $\Longrightarrow$ (2): Assume that for some $e' \geq 1$, we write $F^e_* A$ as

$$F^{e'}_* A = A^{\oplus a_{e'}} \oplus \omega^{\oplus b_{e'}} \oplus M_{e'},$$

where $0 \neq M_{e'}$ is a maximal Cohen-Macaulay A-module that does not have A and ω_A as direct summands. Since R is strongly F-regular by the assumption, as explained in [25, Lemma 3.3] we may now build similar decompositions for $e \geq e'$:

$$F^e_* A = A^{\oplus a_e} \oplus \omega^{\oplus b_e} \oplus M_{e'}^{\oplus c_e} \oplus N_e$$

with $\liminf_{e \to \infty} c_e / \operatorname{rank} F^e_* A > 0$. After taking ranks we then have that

$$1 \geq \frac{a_e}{\operatorname{rank} F^e_* A} + \frac{b_e}{\operatorname{rank} F^e_* A} + \operatorname{rank}_A M_{e'} \frac{c_e}{\operatorname{rank} F^e_* A}$$

which after taking limits then gives that $1 > \mathrm{s}(A) + \mathrm{s}(A)$ which is a contradiction. ☐

Let us give an example of local rings having $\mathrm{s}(A) = \frac{1}{2}$.

Example 4.7. Let $d \geq 2$ be an integer. Let $A = k[[x_1, \ldots, x_d]]^{(2)}$ be the second Veronese subring of the formal power series ring over k. Then $\mathrm{s}(A) = \frac{1}{2}$. Moreover, A is not Gorenstein if and only if d is odd.

Let A be a Cohen-Macaulay local domain with minimal multiplicity. Then A is not Gorenstein if and only if $\mathrm{e}(A) \geq 3$. Moreover, then $\mathrm{type}(A) = \mathrm{e}(A) - 1$. So we can obtain the following corollary by combining 3.6, 4.1, 4.6 and 4.3.

Corollary 4.8. *Suppose that A is a Cohen-Macaulay local domain with minimal multiplicity* $\mathrm{e}(A) \geq 3$. *Then*

(1) $\mathrm{s}(A) \leq \frac{1}{2}$.
(2) $\frac{\mathrm{e}(A)}{2} \leq \mathrm{e_{HK}}(A) \leq (1 - \mathrm{s}(A))\mathrm{e}(A)$.
(3) *The following conditions are equivalent*:
    (a) $\mathrm{s}(A) = \frac{1}{2}$.

(b) *A has FFRT with the FFRT system* $\{A, \omega_A\}$.

When this is the case, $\mathrm{e_{HK}}(A) = \frac{\mathrm{e}(A)}{2}$.

(4) *Suppose* $\mathrm{s}(A) > 0$. *Then the following conditions are equivalent*:

(a) $\mathrm{e_{HK}}(A) = (1 - \mathrm{s}(A))\mathrm{e}(A)$.

(b) *$F^e_* A$ can be written as a direct sum of A, ω_A and Ulrich A-modules for every $e \geq 1$.*

Example 4.9. Let $A = k[[x,y,z]]^{(2)}$. Then A has minimal multiplicity and its multiplcity is 4. Moreover, $\mathrm{e_{HK}}(A) = \frac{\mathrm{e}(A)}{2} = 2$ and $\mathrm{s}(A) = \frac{1}{2}$.

Example 4.10. Let $A = k[[x^3, xy^2, xy^2, y^3]] = k[[x,y]]^{(3)}$. Then, $F^e_* A$ can be written as direct sum of A, $\omega_A = Ax + Ay$ and $M = Ax^2 + Axy + Ay^2$. In this case, $\mathrm{s}(A) = \frac{1}{3}$ by [40], and $\mathrm{e}(A) = 3$, $\mathrm{type}(A) = 2$. Since $\mu_A(M) = 3 = \mathrm{e}_A(M) = \mathrm{e}(A)\,\mathrm{rank}_A(M)$, we see that M is an Ulrich A-module, and we have

$$\mathrm{e_{HK}}(A) = 2 = \mathrm{s}(A)(\mathrm{type}(A) + 1) + 2 \cdot \mathrm{e}(A)\left(\frac{1}{2} - \mathrm{s}(A)\right).$$

We pose the following question.

Question 4.11. Let A be a d-dimensional Cohen-Macaulay local domain with $\mathrm{s}(A) = \frac{1}{2}$. Then is A isomorphic to the ring defined in Example 4.7?

4.1. **$\mathbb{Q}$-Gorenstein local rings.** We are able to give an affirmative answer to Question 4.11 in a particular case.

Let A be a Cohen-Macaulay reduced local ring. For an ideal $I \subset A$ of height 1, the n^{th} symbolic power $I^{(n)}$ denotes the intersection of height one primary components of I^n.

Definition 4.12. Let A be a normal local domain having a canonical module ω_A.

(1) The ring A is said to be $\mathbb{Q}$-*Gorenstein* if there exists an ideal J of height 1 which is isomorphic to ω_A as an A-module such that $J^{(n)}$ is principal. Furthermore,

$$\mathrm{index}(A) := \min\{n \in \mathbb{N} \mid J^{(n)} \text{ is principal}\}$$

is called the *index* of A.

(2) Suppose that A is a $\mathbb{Q}$-Gorenstein normal local domain of $r = \mathrm{index}(A) \geq 2$, and let J be an ideal such that $J \cong \omega_A$. Put

$$B := \bigoplus_{i=0}^{r-1} J^{(i)}.$$

Then B is called the *canonical cover* of A.

Note that A is Gorenstein if and only if it is $\mathbb{Q}$-Gorenstein of index 1.

Definition 4.13. Let $\varphi \colon A \to B$ be ring homomorphism. A ring homomorphism φ is called étale if B is a finitely generated A-algebra and $A \to B$ is flat and unramified. Put

$$U = \{P \in \mathrm{Spec(A)} \mid A_P \to B \otimes_A A_P \text{ is étale}\}.$$

A ring homomorphism φ is called *étale in codimension* 1 if $\mathrm{Spec(A)} \setminus U$ is a closed subset of codimension at least two.

We will need the following result from [37].

Theorem 4.14. *Let A be a $\mathbb{Q}$-Gorenstein strongly F-regular local ring with $r = \mathrm{index}(A) \geq 2$, and B its canonical cover. Suppose that $(r,p) = 1$. Then*

(1) *$A \to B$ is module-finite and étale in codimension* 1.

(2) *The free rank of B as an A-module is one.*

(3) *$A \to B$ is split.*
(4) *B is a strongly F-regular Gorenstein local domain.*

Note that if $A \hookrightarrow B$ is a module-finite extension of local domains, then $\mathrm{e_{HK}}(A) = \frac{1}{[Q(B):\, Q(A)]}\,\mathrm{e_{HK}}(B)$. But such a formula is *not* satisfied in general for F-signatures. So the following theorem is very useful.

Theorem 4.15 ([35, Theorem 2.6.5], see also [29]). *Let $(A,\mathfrak{m},k) \to (B,\mathfrak{m},k)$ be a module-finite extension of normal local domains, where the residue field k has positive characteristic. Suppose that $A \to B$ is split and étale in codimension one and the free rank of B as an A-module is equal to one. Then we have*

$$\mathrm{s}(A) = \frac{1}{[Q(B):\, Q(A)]}\,\mathrm{s}(B).$$

Corollary 4.16. *Let A be a strongly F-regular $\mathbb{Q}$-Gorenstein local domain but not Gorenstein. Let $r \geq 2$ be the index of A such that $(r,p)=1$. Let B be a canonical cover of A. Then we have $\mathrm{s}(A) = \frac{1}{r}\,\mathrm{s}(B)$.*

Using this, we can prove the following theorem.

Theorem 4.17. *Let A be an F-finite $\mathbb{Q}$-Gorenstein normal local domain of characteristic $p>0$. Assume the index r of A satisfies $(r,p)=1$. Then the following conditions are equivalent*:

(1) $\mathrm{s}(A) = \frac{1}{2}$.
(2) *$r=2$ and A admits a canonical cover B which is regular.*

Proof. Suppose (1). Let B a canonical cover of A. Then $\mathrm{s}(A) = \frac{\mathrm{s}(B)}{r}$. If $r \geq 3$, then $\mathrm{s}(B) = r\cdot \mathrm{s}(A) > 1$. This is a contradiction. Hence $r=2$ and $\mathrm{s}(B)=1$. Hence B is regular. The converse is easy. □

4.2. **The F-signatures of 3-dimensional Gorenstein rings.** We want to present a few upper bounds on F-signature in view of Question 1.1. We first estimate F-signature using the multiplicity.

Theorem 4.18. *Let $(A,\mathfrak{m},k)$ be a 3-dimensional Gorenstein F-regular local ring with multiplicity $\mathrm{e}(A) \geq 3$. Then $\mathrm{s}(A) \leq \frac{\mathrm{e}(A)}{24}$.*

Proof. Let J be a minimal reduction of $\mathfrak{m}$. Then we can write $J:\,\mathfrak{m} = (J,u)$ for some $u \in \mathfrak{m}\setminus J$ because A is Gorenstein. Moreover, we have

$$\mathrm{s}(A) \leq \mathrm{e_{HK}}(J) - \mathrm{e_{HK}}(J:\,\mathfrak{m}) = \mathrm{e}(J) - \mathrm{e_{HK}}(J:\,\mathfrak{m}) = \mathrm{e} - \mathrm{e_{HK}}(J:\,\mathfrak{m}).$$

Since A is F-regular, the Briançon–Skoda theorem implies that $\mathfrak{m}^3 \subset J$, and thus $\mathfrak{m}^2 \subset J:\,\mathfrak{m}$. Since A is not double point, $\mathfrak{m}^2 \not\subset J$. Hence there exists an element $v \in \mathfrak{m}^2$ such that $v \in J:\,\mathfrak{m}\setminus J$. Write $v = a + ru$ for some $a \in J$ and $r \in A$. Suppose $r \in \mathfrak{m}$. Then $ru \in J$ and thus $v = a+ru \in J$, which is a contradiction. Hence $r \in A\setminus\mathfrak{m}$ and $(J,u) = (J,v)$. So we may assume that $u \in \mathfrak{m}^2$. Then $u^q \in \mathfrak{m}^{2q}$ and we get

$$\begin{aligned}\ell_A\left(\frac{u^qA + J^{[q]}}{J^{[q]}}\right) &\leq \ell_A\left(\frac{u^qA + \mathfrak{m}^{\frac{5}{2}q} + J^{[q]}}{J^{[q]}}\right)\\ &= \ell_A\left(\frac{u^qA + \mathfrak{m}^{\frac{5}{2}q} + J^{[q]}}{\mathfrak{m}^{\frac{5}{2}q} + J^{[q]}}\right) + \ell_A\left(\frac{\mathfrak{m}^{\frac{5}{2}q} + J^{[q]}}{J^{[q]}}\right).\end{aligned}$$

Note that $\overline{A} = A/J^{[q]}$ is an Artinian Gorenstein local ring. Thus the Matlis duality yields that

$$\ell_A\left(\frac{u^qA + \mathfrak{m}^{\frac{5}{2}q} + J^{[q]}}{\mathfrak{m}^{\frac{5}{2}q} + J^{[q]}}\right) \leq \ell_A(A/\mathfrak{m}^{\frac{1}{2}q}) \quad \text{and} \quad \ell_A\left(\frac{\mathfrak{m}^{\frac{5}{2}q} + J^{[q]}}{J^{[q]}}\right) \leq \ell_A(A/\mathfrak{m}^{\frac{1}{2}q}).$$

Therefore

$$\mathrm{s}(A) = \lim_{e\to\infty} \ell_A\left(\frac{u^qA + J^{[q]}}{J^{[q]}}\right)/q^d \leq 2\cdot \lim_{e\to\infty}\frac{\ell_A(A/\mathfrak{m}^{\frac{1}{2}q})}{q^d} = 2\times\frac{1}{3!}\left(\frac{1}{2}\right)^3\mathrm{e}(A) = \frac{1}{24}\mathrm{e}(A),$$

as required. □

The next example shows Theorem 4.18 gives the best possible bound.

Example 4.19. Let $R^{(2)}$ be the 2^{nd} Veronese subring of $R = k[x,y,z,w]/(xw-yz)$. Set

$$A = k[[(x,y,z,w)^2]]/(xw-yz),$$

which is the completion with respect to the irrelevant maximal ideal $\mathfrak{m}$ of $R^{(2)}$. Then A is a 3-dimensional Gorenstein F-regular local domain. Hence [19, Theorem 5.1(1)] implies $\mathrm{e}(A) \le \mathrm{emb}(A) - 1 = 8$. On the other hand, since A is not hypersurface (of multiplicity 2), $\mathrm{e}(A) \ge \mathrm{emb}(A) - \dim A + 2 = 9 - 3 + 2 = 8$ and thus $\mathrm{e}(A) = 8$. Moreover, we have

$$\mathrm{s}(A) = \frac{\mathrm{s}(R)}{2} = \frac{2/3}{2} = \frac{1}{3} = \frac{\mathrm{e}(A)}{24}.$$

On the other hand, by [39, Corollary 1.10], we have

$$\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}_R^2) = \mathrm{e}(\mathfrak{m})\binom{2+3-2}{3} + \mathrm{e}_{\mathrm{HK}}(\mathfrak{m})\binom{2+3-2}{3-1} = \mathrm{e}(R) + 3\cdot \mathrm{e}_{\mathrm{HK}}(R) = 2 + 3\cdot\frac{4}{3} = 6.$$

Hence

$$\mathrm{e}_{\mathrm{HK}}(A) = \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}_A R)}{2} = \frac{\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}_R^2)}{2} = \frac{6}{2} = 3 > \frac{7}{3} = \frac{8}{6} + 1 = \frac{\mathrm{e}(A)}{6} + 1.$$

Remark 4.20. The argument given in the proof of Theorem 4.18 is also valid for some classes of higher dimensional Gorenstein rings. For example, let $A := k[[x_0, x_1, \dots, x_d]]/(x_0^d + x_1^d + \cdots + x_d^d)$. For the maximal ideal $\mathfrak{m}$ and its minimal reduction J, we have that $\mathfrak{m}^d \subset J$ and $\mathfrak{m}^{d-1} \not\subset J$. Thus, by the same argument as the proof of Theorem 4.18, we see that $\mathrm{s}(A) \le \frac{\mathrm{e}(A)}{2^{d-1}d!} = \frac{1}{2^{d-1}(d-1)!}$, see also [40, Proposition 2.4 and Question 2.6].

As the first open case, we will investigate Question 1.1, Conjecture 2.10 for 3-dimensional Gorenstein rings. In particular, we ask the following question.

Question 4.21. Let $(A,\mathfrak{m})$ be a 3-dimensional non-regular Cohen-Macaulay local ring. Is $\mathrm{s}(A) \le \frac{2}{3}$?

If this is correct, then this bound is best possible because if $A = k[x,y,z,w]/(xw-yz)$, then $\mathrm{s}(A) = \frac{2}{3}$. We will give a positive answer to this question for the case of toric rings in the next section (see Theorem 5.9). For a general situation, we only have the inequality given in Proposition 4.22.

Proposition 4.22. *Let A be a 3-dimensional F-regular local domain which is not regular. Then* $\mathrm{s}(A) < \frac{5}{6}$.

Proof. We may assume that A is Gorenstein (see Proposition 4.1). By Proposition 4.4(1), we have

$$\mathrm{e}_{\mathrm{HK}}(A) \le \mathrm{s}(A) + \mathrm{e}(A)(1 - \mathrm{s}(A)).$$

On the other hand, Theorem 3.7 implies that

$$\frac{\mathrm{e}(A)}{6} + 1 \le \mathrm{e}_{\mathrm{HK}}(A) \le \mathrm{s}(A) + \mathrm{e}(A)(1 - \mathrm{s}(A)).$$

Hence

$$\mathrm{e}(A)\left(\mathrm{s}(A) - \frac{5}{6}\right) \le \mathrm{s}(A) - 1 < 0.$$

and thus $\mathrm{s}(A) < \frac{5}{6}$. □

5. Observations on toric rings

In this section, we further study an upper bound on F-signature of a toric ring. In particular, in Theorem 5.9 we give a positive answer to Question 4.21.

5.1. **Preliminaries.** Let $\mathsf{N} \cong \mathbb{Z}^d$ be a lattice of rank d. Let $\mathsf{M} = \mathrm{Hom}_{\mathbb{Z}}(\mathsf{N}, \mathbb{Z})$ be the dual lattice of N. We set $\mathsf{N}_{\mathbb{R}} = \mathsf{N} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathsf{M}_{\mathbb{R}} = \mathsf{M} \otimes_{\mathbb{Z}} \mathbb{R}$. We denote the inner product by $\langle \quad , \quad \rangle \colon \mathsf{M}_{\mathbb{R}} \times \mathsf{N}_{\mathbb{R}} \to \mathbb{R}$. Let

$$\sigma := \mathbb{R}_{\geq 0} v_1 + \cdots + \mathbb{R}_{\geq 0} v_n \subset \mathsf{N}_{\mathbb{R}}$$

be a strongly convex rational polyhedral cone of dimension d generated by $v_1, \ldots, v_n \in \mathbb{Z}^d$ where $d \leq n$. We assume that $v_1, \ldots, v_n$ are minimal generators of σ. For each generator, we define the linear form $\lambda_i(-) := \langle -, v_i \rangle$. We consider the dual cone $\sigma^\vee$:

$$\sigma^\vee := \{\mathbf{x} \in \mathsf{M}_{\mathbb{R}} \mid \lambda_i(\mathbf{x}) \geq 0 \text{ for all } i = 1, 2, \ldots, n\}.$$

In this case, $\sigma^\vee \cap \mathsf{M}$ is a positive normal affine monoid. Given an algebraically closed field k of characteristic $p > 0$, we define the toric ring

$$A := k[\sigma^\vee \cap \mathsf{M}] = k[t_1^{m_1} \cdots t_d^{m_d} \mid (m_1, \ldots, m_d) \in \sigma^\vee \cap \mathsf{M}].$$

We denote the irrelevant ideal of A as $\mathfrak{m}$.

For each $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we set

$$V(\mathbf{a}) := \{\mathbf{x} \in \mathsf{M} \mid (\lambda_1(\mathbf{x}), \ldots, \lambda_n(\mathbf{x})) \geq (a_1, \ldots, a_n)\}.$$

Then we define the divisorial ideal (rank one reflexive module) $D(\mathbf{a})$ generated by all monomials whose exponent vectors are in $V(\mathbf{a})$. For example, we have that $R = D(\mathbf{0})$ and $\omega_A \cong D(1, 1, \ldots, 1)$. Let $\mathfrak{p}_i := D(\delta_{i1}, \ldots, \delta_{in})$, where δ_{ij} is the Kronecker delta. The height one prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ generate the class group $\mathrm{Cl}(A)$. When we consider a divisorial ideal $D(\mathbf{a})$ as the element of $\mathrm{Cl}(A)$, we denote it by $[D(\mathbf{a})]$.

In what follows, we will pay attention to a certain class of divisorial ideals called conic.

Definition 5.1 (see e.g. [7, 6])**.** We say that a divisorial ideal $D(\mathbf{a})$ is *conic* if there exist $\mathbf{x} \in \mathsf{M}_{\mathbb{R}}$ such that $\mathbf{a} = (\ulcorner \lambda_1(\mathbf{x}) \urcorner, \ldots, \ulcorner \lambda_n(\mathbf{x}) \urcorner)$, where $\ulcorner \quad \urcorner$ stands for the round up.

Any conic divisorial ideal is a rank one MCM module (see [7, Corollary 3.3]). We denote the set of isomorphism classes of conic divisorial ideals of a toric ring A by $\mathscr{C}(A)$. This is a finite set because the number of isomorphism classes of rank one MCM A-modules is finite (see [7, Corollary 5.2]). The following proposition guarantees that any conic divisorial ideal appears in $F_*^e A$ as a direct summand.

Theorem 5.2 ([7, Proposition 3.6], [32, Subsection 3.2])**.** *Let A be a toric ring as above. Then, A has FFRT by the FFRT system $\mathscr{C}(A)$.*

We recall that our arguments can be reduced to the $\mathfrak{m}$-adic completion of A, as we mentioned in the beginning of Section 4. Thus, we may assume that A is complete local, in which case the Krull–Schmidt condition holds for A.

The F-signature of a toric ring can be computed combinatorially and, in particular, does not depend on the characteristic.

Theorem 5.3 ([40, Theorem 5.1], see also [6, 31, 35])**.** *Let A be a toric ring. Then, we may compute*

$$\mathrm{s}(A) = \mathrm{vol}\{\mathbf{x} \in \mathsf{M}_{\mathbb{R}} \mid 0 \leq \lambda_i(\mathbf{x}) \leq 1 \textit{ for all } i\}.$$

Remark 5.4. In some parts of this section, we assume that if the class group $\mathrm{Cl}(A)$ contains a torsion element, then the order of that element is coprime to p. In this case, the toric ring A is a ring of invariants.

Namely, let $k^\times$ be the multiplicative group of k and $G := \mathrm{Hom}(\mathrm{Cl}(A), k^\times)$ be the character group of $\mathrm{Cl}(A)$. The group G acts on $B := k[x_1, \ldots, x_n]$ by $g \cdot x_i = g([\mathfrak{p}_i]) x_i$ for each $g \in G$ and any i. Then, by [7, Theorem 2.1(b)], A can be described as $A \cong B^G$. Moreover, to avoid the triviality, we assume that $g([\mathfrak{p}_i]) \neq 1$ for any i, that is, $[\mathfrak{p}_i] \neq 0$ in $\mathrm{Cl}(A)$.

5.2. **Cohen-Macaulay toric rings.** We recall that the F-signature of non-Gorenstein ring is less than or equal to $\frac{1}{2}$ (see Propsition 4.1). We now determine toric rings whose F-signature is $\frac{1}{2}$.

Proposition 5.5. *Let A be a toric ring as in Remark 5.4. Then, the following conditions are equivalent.*

(1) *The FFRT system of A is $\{A, M\}$ with $M \ncong A$.*
(2) *A is isomorphic to the Veronese subring $k[x_1,\dots,x_d]^{(2)}$ of degree 2.*

When this is the case, the F-signature is $\mathrm{s}(A) = \frac{1}{2}$.

Proof. We first assume that the FFRT system of A is $\{A, M\}$. We note that $M \cong \omega_A$ if A is not Gorenstein. In fact, ω_A certainly appears in $F^e_* A$ as a direct summand for sufficiently large e, if A is strongly F-regular (cf. [30, Proof of Proposition 3.10], [15, Proposition 2.1]). By [7, Remark 3.4], the divisorial ideals $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ are conic. Since $\mathscr{C}(A) = \{A, M\}$, we have that $\mathfrak{p}_1 \cong \cdots \cong \mathfrak{p}_n \cong M$. Thus we have that $[\mathfrak{p}_1] = \cdots = [\mathfrak{p}_n] = [M]$ in $\mathrm{Cl}(A)$. On the other hand, since $[\omega_A] = [\mathfrak{p}_1] + \cdots + [\mathfrak{p}_n]$, we see that $n[\mathfrak{p}_i] = 0$ (resp. $(n-1)[\mathfrak{p}_i] = 0$) in $\mathrm{Cl}(A)$ for any i if A is Gorenstein (resp. not Gorenstein). Thus we conclude that $\mathrm{Cl}(A)$ is a finite cyclic group generated by $[\mathfrak{p}_i]$, that is, $\mathrm{Cl}(A) \cong \langle [\mathfrak{p}_i] \rangle$. This implies that the cone σ defining A is simplicial (i.e., $n = d$), thus we have that $A \cong B^G$ where $B = k[x_1,\dots,x_d]$ and $G = \mathrm{Hom}(\mathrm{Cl}(A), k^\times)$ is a finite cyclic group. We may assume that G is small (see e.g. [20, Proof of Theorem 5.7]). By [32, Proposition 3.2], each indecomposable direct summand of $F^e_* A$ is a module of covariants which takes the form $(B \otimes_k V_i)^G$, where V_i is an irreducible representation of G. Since the FFRT system is $\{A, M\}$ and G is small, we have only two non-isomorphic irreducible representations of G. Then we have that $|G| = 2$, and the elements of G are the characters of $\rho_0, \rho_1 \in \mathrm{Hom}(\mathrm{Cl}(A), k^\times)$ defined by $\rho([\mathfrak{p}_i]) = 1$ and $\rho([\mathfrak{p}_i]) = -1$ respectively. Consequently, we have that $A \cong k[x_1,\dots,x_d]^{(2)}$. By [40, Theorem 4.2], the F-signature of $A \cong B^G$ is $\frac{1}{|G|} = \frac{1}{2}$.

On the other hand, we assume that $A \cong k[x_1,\dots,x_d]^{(2)}$. Then, A is the invariant subring of $k[x_1,\dots,x_d]$ under the action of the cyclic group $\langle g = \mathrm{diag}(-1,\dots,-1) \rangle$ defined by $g \cdot x_i = -x_i$ for any i. Thus, the condition (1) follows from [32, Proposition 3.2.1]. □

This is the main result in this subsection.

Theorem 5.6. *Let A be a toric ring as Remark 5.4. Assume that A is not Gorenstein, then the following conditions are equivalent.*

(1) $\mathrm{s}(A) = \frac{1}{2}$.
(2) *The FFRT system of A is $\{A, \omega_A\}$.*
(3) *A is isomorphic to the Veronese subring $k[x_1,\dots,x_d]^{(2)}$ of degree 2, where d is an odd number.*

Proof. (3) $\Longrightarrow$ (1) follows from [40, Theorem 4.2]. (1) $\Longleftrightarrow$ (2) follows from Theorem 4.6. Then we show (2) $\Longleftrightarrow$ (3). By Proposition 5.5, we see that $A \cong k[x_1,\dots,x_d]^{(2)} = k[x_1,\dots,x_d]^G$, where $G \cong \langle \mathrm{diag}(-1,\dots,-1) \rangle$. Since A is not Gorenstein, G is not a subgroup of $SL(d,k)$, thus d is an odd number (see [36]). □

5.3. **Gorenstein toric rings.** We now switch our attention to Gorenstein toric rings. First, we remark that if A is a Gorenstein toric ring, then $\mathrm{s}(A) = \frac{1}{2}$ does not imply the conditions (2) and (3) in Theorem 5.6 (see Example 5.7 below), although we have an equivalence as in Proposition 5.5 which induces $\mathrm{s}(A) = \frac{1}{2}$.

Example 5.7. We consider the Segre product $P_n := k[x_1,y_1] \# \cdots \# k[x_n,y_n]$ of n polynomial rings with two variables, which is a Gorenstein toric ring in dimension $n+1$. Then, by [15, Proposition 6.1], one can compute $\mathrm{s}(P_n) = \frac{2}{n+1}$. For example, $\mathrm{s}(P_3) = \frac{1}{2}$ but the FFRT system of P_3 consists of 7 conic divisorial ideals (see [14, Example 2.6]).

Furthermore, the F-signature of a Gorenstein toric ring can be greater than $\frac{1}{2}$. For example, $\mathrm{s}(P_2) = \frac{2}{3}$ or if $A = k[x_1,y_1,z_1] \# k[x_2,y_2,z_2]$, a Gorenstein toric ring of dimension 5, then $\mathrm{s}(A) = \frac{66}{120} = \frac{11}{20}$ (see e.g.

[40, Theorem 5.8], [31, Example 7], [15, Proposition 6.2]), and the FFRT system consists of 5 conic divisorial ideals (see [14, Example 2.6]).

As these example show, when A is Gorenstein it is difficult to bound F-signature using the number of modules in the FFRT system. For the reference, we also give the observation regarding Gorenstein toric rings whose the FFRT system consists of three modules.

Proposition 5.8. *Let A be a toric ring as in Remark 5.4. We assume that A is Gorenstein. Then, the following conditions are equivalent.*

(1) *The FFRT system of A is $\{A, M_1, M_2\}$,*
(2) *A is isomorphic to one of the following rings:*
 (a) *the invariant subring $k[x_1,\dots,x_d]^G$ where $G = \langle \mathrm{diag}(\underbrace{\xi,\dots,\xi}_{m},\underbrace{\xi^2,\dots,\xi^2}_{m})\rangle$ with $d = 2m$ and ξ is a primitive cubic root of unity, in which case $\mathrm{s}(A) = \frac{1}{3}$,*
 (b) *the Segre product $k[x_1,y_1]\#k[x_2,y_2] = k[x_1x_2, x_1y_2, y_1x_2, y_1y_2]$ of two polynomial rings, in which case $\mathrm{s}(A) = \frac{2}{3}$.*

Proof. We first show $(1) \Longrightarrow (2)$. By [15, Proposition 2.1], we have that $M_1 \cong M_2^*$. Since $\mathfrak{p}_i$ is conic, each $\mathfrak{p}_i$ is isomorphic to either M_1 or M_1^*. Moreover, since $[\omega_A] = [\mathfrak{p}_1] + \cdots + [\mathfrak{p}_n] = 0$, we see that n is an even number and we may assume that

$$[\mathfrak{p}_1] = \cdots = [\mathfrak{p}_m] = -[\mathfrak{p}_{m+1}] = \cdots = -[\mathfrak{p}_n]$$

where $n = 2m$. Then, we see that $\mathrm{Cl}(A)$ is generated by $[\mathfrak{p}_1]$, and we have two cases depending on whether it is torsion.

- If $\mathrm{Cl}(A) \cong \mathbb{Z}/r\mathbb{Z}$, then, by an argument similar to the proof of Proposition 5.5, $G \cong \mathbb{Z}/3\mathbb{Z}$. For a generator g of G, we can set $g([\mathfrak{p}_1]) = \xi$ where ξ is a primitive cubic root of unity. In this case, the action of G in S can be described as

$$\begin{cases} g\cdot x_i = \xi x_i & (i = 1,\dots,m), \\ g\cdot x_i = \xi^{-1}x_i = \xi^2 x_i & (i = m+1,\dots,n), \end{cases}$$

 and we have the case (a). The F-signature of A can be obtained from [40, Theorem 4.2].

- If $\mathrm{Cl}(A) \cong \mathbb{Z}$, then we see that $G \cong k^\times$ and if $g([\mathfrak{p}_1]) = \zeta \in k^\times$ for a generator g of G, then $g(-[\mathfrak{p}_1]) = \zeta^{-1}$. Thus, the action of G on B can be described as

$$\begin{cases} g\cdot x_i = \zeta x_i & (i = 1,\dots,m), \\ g\cdot x_i = \zeta^{-1}x_i & (i = m+1,\dots,n). \end{cases}$$

 In this case, we have that

$$A \cong k[x_1,\dots,x_m]\#k[x_{m+1},\dots,x_n] = k[x_ix_j \mid i = 1,\dots,m,\ j = m+1,\dots,n].$$

 This Segre product of two polynomial rings can be considered as a Hibi ring [13], and the conic classes in Hibi rings are characterized in [14]. By [14, Theorem 2.4 and Example 2.6], we see that the Segre products of two polynomial rings that satisfy the condition (1) are only the one with $m = 2$. Thus, we have that $\mathrm{s}(A) = \frac{2}{3}$ by Example 5.7 (the case of $n = 2$), or we easily see that $A \cong k[x,y,z,w]/(xw - yz)$, in which case $\mathrm{s}(A) = \mathrm{e}(A) - \mathrm{e}_{\mathrm{HK}}(A) = 2 - \frac{4}{3} = \frac{2}{3}$.

$(2) \Longrightarrow (1)$ is well known, see e.g. [32, Proposition 3.2] for the case (a) and [33, the proof of Theorem 6.1] for (b). □

In the rest, we further assume that $d = \dim A = 3$ and A is Gorenstein. In this case, it is known that we can take minimal generators $v_1, \dots, v_n$ of σ as $v_i := (v'_i, 1)$ where $v'_i \in \mathbb{Z}^2$ for $i = 1, \dots, n$. Therefore, the convex hull of $v'_1, \dots, v'_n$ forms a lattice polygon $\Delta_A \subset \mathbb{R}^2$, called the *toric diagram* of A. We note that parallel translations of Δ_A in $\mathbb{R}^2$ do not change the associated toric ring. Namely, if $\widetilde{\Delta} \subset \mathbb{R}^2$ is a lattice polygon obtained by applying a parallel translation to Δ_A and $\widetilde{\sigma} \subset \mathbb{R}^3$ is the cone determined by putting $\widetilde{\Delta}$ on the hyperplane at height one, then σ and $\widetilde{\sigma}$ are unimodular equivalent, and hence $A = k[\sigma^\vee \cap \mathbb{Z}^3] \cong k[\widetilde{\sigma}^\vee \cap \mathbb{Z}^3]$. Thus, it is only the shape of Δ_A that matters.

We then have the following theorem that gives the affirmative answer to Question 4.21.

Theorem 5.9. *Let A be a 3-dimensional non-regular Gorenstein toric ring as above. The following conditions are equivalent;*

(1) $\mathrm{s}(A) > \frac{1}{2}$,
(2) *A is isomorphic to $k[x,y,z,w]/(xy - zw)$.*

When this is the case, we have that $\mathrm{s}(A) = \frac{2}{3}$.

Proof. We first assume that $A \cong k[x,y,z,w]/(xy - zw)$. We already know that $\mathrm{s}(A) = \frac{2}{3}$, thus (1) holds.

To show (1)⇒(2), we consider the cone $\sigma = \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_n \subset \mathbb{R}^3$ that gives the toric ring A. We choose three vectors $v_\alpha, v_\beta, v_\gamma$ where $\alpha, \beta, \gamma \in \{1, \dots, n\}$. Let σ' be the cone generated by $v_\alpha, v_\beta, v_\gamma$, and let $A' := k[(\sigma')^\vee \cap \mathbb{Z}^3]$. The F-signature is the volume of the cell determined by the linear forms $\lambda_i(-) = \langle -, v_i \rangle$ as shown in Theorem 5.3, thus we have that $\mathrm{s}(A) \leq \mathrm{s}(A')$ by the construction of A'. Here, the cone σ' is simplicial, thus $A' \cong k[X,Y,Z]^G$ for some abelian group $G \subset \mathrm{SL}_3(k)$ if the characteristic of k is sufficiently large. By Theorem 5.3, we see that the F-signature of a toric rings does not depend on the characteristic, thus when we consider the F-signature, we may assume that $A' \cong k[X,Y,Z]^G$, in which case $\mathrm{s}(A) = \frac{1}{|G|}$ (see [40, Theorem 4.2]). On the other hand, it is known that $|G|$ is equal to the number of elementary triangles contained in $\Delta_{A'}$. Therefore, if $\Delta_{A'}$ is not an elementary triangle, then $\mathrm{s}(A') \leq \frac{1}{2}$.

We now turn our attention to Δ_A again. If the boundary of Δ_A contains a lattice point except the vertices (i.e., A is not an isolated singularity), then we can choose $v_\alpha, v_\beta, v_\gamma$ so that the boundary of $\Delta_{A'}$ contains a lattice point except the vertices. Also, if Δ_A contains an interior lattice point, then we can choose $v_\alpha, v_\beta, v_\gamma$ so that $\Delta_{A'}$ contains an interior lattice point of Δ_A (such a point might be the boundary of $\Delta_{A'}$). In any case, $\Delta_{A'}$ is not an elementary triangle, and hence $\mathrm{s}(A) \leq \frac{1}{2}$. As a conclusion, we see that if $\mathrm{s}(A) > \frac{1}{2}$, then Δ_A does not contain an interior lattice point and a boundary lattice point except the vertices. Thus, Δ_A is unimodular equivalent to the convex hull of $\{(0,0),(1,0),(0,1)\}$ or that of $\{(0,0),(1,0),(1,1),(0,1)\}$ (see e.g., [27, Theorem 1]). If Δ_A is the former one, then A is regular, thus we may assume that A is given by the cone generated by $\{(0,0,1),(1,0,1),(1,1,1),(0,1,1)\}$. In this case, we easily verify that $A \cong k[x,y,z,w]/(xy - zw)$, thus we have (1)⇒(2). □

By Theorem 5.9, we have the upper bound $\frac{2}{3}$ of the F-signatures for 3-dimensional non-regular toric rings. As we showed in Example 5.7, the F-signatures of some higher dimensional Gorenstein toric rings exceed $\frac{1}{2}$, but they are not greater than $\frac{2}{3}$. Thus, we propose the following question.

Question 5.10. Let A be a non-regular toric ring with isolated singularities. Is it true that $\mathrm{s}(A) \leq \frac{2}{3}$?

If A is not a toric ring, we have a family of 3-dimensional Gorenstein rings whose F-signature are greater than $\frac{1}{2}$.

Example 5.11. Let $c > 2$ be an integer. Put $A := k[[x,y,z,w]]/(x^2 + y^2 + z^2 + w^c)$, where k is an algebraically closed field of characteristic $p > c$. Then, A is a normal hypersurface of $\dim A = 3$, and

$$\frac{1}{2} < \mathrm{s}(A) < \frac{2}{3}.$$

In fact, since $\mathrm{e}(A) = 2$ we have that $\mathrm{s}(A) = 2 - \mathrm{e}_{\mathrm{HK}}(A)$, thus this follows from [41, Corollary 3.11].

We remark that if the characteristic of k is zero, the rings discussed in Theorem 5.9 and Example 5.11 have terminal singularities and they play an important role in the Minimal Model Program. Thus, these results suggest that these type of rings also have a nice property from the viewpoint of F-singularities.

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REFERENCES

[1] Ian M. Aberbach and Florian Enescu, *Lower bounds for Hilbert-Kunz multiplicities in local rings of fixed dimension*, Michigan Math. J. **57** (2008), 1–16.
[2] Ian M. Aberbach and Florian Enescu, *New estimates of Hilbert-Kunz multiplicities for local rings of fixed dimension*, Nagoya Math. J. **212** (2013), 59–85.
[3] Ian M. Aberbach and Graham Leuschke, *The F-signature and strongly F-regularity*, Math. Res. Lett. **10** (2003), 51–56.
[4] Manuel Blickle and Frorian Enescu, *On rings with small Hilbert-Kunz multiplicity*, Prco. Amer. Math. Soc. **132** (2004), 2505–2509.
[5] Joseph P. Brennan, Jürgen Herzog, and Bernd Ulrich, *Maximally generated Cohen-Macaulay modules*, Math. Scand. **61** (1987), 181–203.
[6] Winfried Bruns, *Conic divisor classes over a normal monoid algebra*, Commutative Algebra and algebraic geometry, Contemp. Math., **390**, Amer. Math. Soc. (2005), 63–71.
[7] Winfried Bruns and Joseph Gubeladze, *Divisorial linear algebra of normal semigroup rings*, Algebra and Represent. Theory, **6** (2003), 139–168.
[8] Frorian Enescu and Kazuma Shimomoto, *On the upper semi-continuity of the Hilbert-Kunz multiplicity*, J. Algebra **285** (2005), 222–237.
[9] O. Gabber, *Notes on some t-structures*, Geometric Aspects of Dwork Theory. Vol. II, Walter de Gruyter GmbH & Co. KG (2004),711–734.
[10] Ira M. Gessel and Paul Monsky, *The limit as $p \to \infty$ of the Hilbert-Kunz multiplicity of $\sum_{i=1}^{d} x_i^{d_i}$*, arXiv:1007.2004.
[11] C. Han and Paul Monsky, *Some surprising Hilbert-Kunz functions*, Math. Z. **214**, 119–135.
[12] Douglas Hanes, *Notes on the Hilbert-Kunz function*, J. Algebra **265** (2003), 619–630.
[13] Takayuki Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, In:"Commutative Algebra and Combinatorics" (M. Nagata and H. Matsumura, Eds.), Adv. Stud. *Pure Math.* **11**, North–Holland, Amsterdam, (1987), 93–109.
[14] Akihiro Higashitani and Yusuke Nakajima, *Conic divisorial ideals of Hibi rings and their applications to non-commutative crepant resolutions*, Selecta Math. (N.S.), **25** (2019), no. 78.
[15] Akihiro Higashitani and Yusuke Nakajima, *Generalized F-signatures of Hibi rings*, arXiv:1911.08720.
[16] Melvin Hochster and Craig Huneke, *Tight closure, invariant theory, and the Briancon-Skoda theorem*, J. Amer. Math. Soc. **3** (1990), 31–116.
[17] Melvin Hochster and Craig Huneke, *F-regularity, test elements, and smooth base change*, Trans. Amer. Math. Soc. **346** (1994), 1–62.
[18] Craig Huneke and Graham Leuschke, *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. **324** (2002), no. 2, 391–404.
[19] Craig Huneke and Kei-ichi Watanabe, *Upper bound of multiplicity of F-pure rings*. Proc. Amer. Math. Soc. **143** (2015), no. 12, 5021–5026.
[20] Osamu Iyama and Michael Wemyss, *Maximal modifications and Auslander-Reiten duality for non-isolated singularity*, Invent. Math. **197** (2014), 521–586.
[21] Ernst Kunz, *Characterizations of regular local rings of characteristic p*, Amer. J. Math. **41** (1969), 772–784.
[22] Ernst Kunz, *On Noetherian rings of characteristic p*, Amer. J. Math. **98** (1976), 999–1013.
[23] Linquan Ma, Thomas Polstra, Karl Schwede, Kevin Tucker, *F-signature under birational morphisms*, Forum Math. Sigma **7** (2019), e11, 20 pp.
[24] Paul Monsky, *The Hilbert-Kunz function*, Math. Ann. **263** (1983), 43–49.
[25] Thomas Polstra, Ilya Smirnov, *Equimultiplicity theory of strongly F-regular rings*, Mich. Math. J. to appear.

[26] Thomas Polstra, Kevin Tucker, *F-signature and Hilbert-Kunz multiplicity: a combined approach and comparison*, Algebra Number Theory **12** (2018), no. 1, 61–97.
[27] S. Rabinowitz, *A census of convex lattice polygons with at most one interior lattice point*, Ars Combin. **28** (1989), 83–96.
[28] David Rees, *Valuations associated with ideals, II*, J. London Math. Soc. **31** (1956), 221–228.
[29] Javier Carvajal-Rojas, Karl Schwede, Kevin Tucker, *Fundamental groups of F-regular singularities via F-signature*, Ann. Sci. Éc. Norm. Supér. (4) **51** (2018), no. 4, 993–1016.
[30] Akiyoshi Sannai, *On Dual F-Signature*, Int. Math. Res. Not. IMRN, Vol. 2015, no.1, pp. 197–211.
[31] Anurag K. Singh, *The F-signature of an affine semigroup ring*, J. Pure Appl. Algebra **196** (2005), no. 2-3, 313–321.
[32] Karen E. Smith, Michel Van den Bergh, *Simplicity of Rings of Differential Operators in Prime Characteristic*, Proc. London Math. Soc. (3) **75** (1997), 32–62.
[33] Yukinobu Toda and Takehiko Yasuda, *Noncommutative resolution, F-blowups and D-modules*, Adv. Math. **222** (2009), no. 1, 318–330.
[34] Kevin Tucker, *F-signature exists*, Invent. Math. **190** (2012), 743–765.
[35] Michael R. Von Korff, *The F-signature of Toric Varieties*, the dissertation in the University of Michigan, 2012.
[36] Kei-ichi Watanabe, *Certain invariant subrings are Gorenstein*, Osaka, J. Math. **71**, (1974), 1–8.
[37] Kei-ichi Watanabe, *F-regular and F-pure normal graded rings*, J. Pure Appl. Algebra **71**, (1991), 341–350.
[38] Kei-ichi Watanabe and Ken-ichi Yoshida, *Hilbert-Kunz multiplicity and an inequality between multiplicity and colength*, J. Algebra **230**, (2000), 295–317.
[39] Kei-ichi Watanabe and Ken-ichi Yoshida, *Hilbert-Kunz multiplicity of two-dimensional local rings*, Nagoya Math. J. **162** (2001), 87–110.
[40] Kei-ichi Watanabe and Ken-ichi Yoshida, *Minimal relative Hilbert-Kunz multiplicity*, Illinois J. Math. J. **48** (2004), 273–294.
[41] Kei-ichi Watanabe and Ken-ichi Yoshida, *Hilbert-Kunz multiplicity of three-dimensional local rings*, Nagoya Math. J. **177** (2005), 47–75.
[42] Yongwei Yao, *Modules with Finite F-Representation Type*, J. London Math. Soc. **72** (2005), no. 2, 53–72.
[43] Yongwei Yao, *Observations on the F-signature of local rings of characteristic p*, J. Algebra **299** (2006) no. 1, 198–218.
[44] Ken-Ichi Yoshida, *Small Hilbert-Kunz multiplicity and (A_1)-type singularity* in Proceedings of the 4th Japan-Vietnam Joint Seminar on Commutative Algebra by and for Young Mathematicians, Meiji University, Japan, 2009.

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