How complicated are polynomials? Andrew Snowden

First, let's consider polynomials in a fixed number n

$R = \mathbb{C}[x_1, \cdots, x_n]$

Two important problems:

① Solving linear equations

② Solving nonlinear equations

Important Theorem: All solutions are finitely generated.

$\Downarrow$

Hilbert Basis Theorem: $\mathbb{C}[x_1, \cdots, x_n]$ is Noetherian.

Two more finite results:

① Suppose $I = (f_1, \cdots, f_k)$ and all generators are homogeneous

THM: Hilbert Series $H_I(t) = \sum_{m \geq 0} \dim(I_m) \cdot t^m$ is a rational function

② Free resolution of I

$$\cdots \to R^{r_1} \to R^{r_0} \to I$$

THM (Hilbert Syzygy THM) $\mathrm{pd}_R I \leq n$

We now look at $n \to +\infty$

Stillman [2000]: If we fix d, r, and consider r polynomials of degree at most d, then the complexity is bounded as $n \to +\infty$.

Stillman's Conjecture: Fix r, d, then $\exists B(r,d)$ s.t. for any ideal $I = (f_1, \cdots, f_r)$ with $\deg f_i \leq d$, $\mathrm{pd}_R I \leq B$.

Anoyan - Hochster [2016]

Let F be a homogeneous polynomial. Then the strength of F is the minimal number s s.t. $F = \sum_{i=1}^{s} G_i H_i$ where G_i, H_i are homogeneous of smaller degree

EX: ① $X_1^4 + X_2^4$: strength is at most 2

② $X_1^2 - X_2^2 = (X_1 + X_2)(X_1 - X_2)$ strength is 1

③ If F is a polynomial in $x_1, \cdots, x_n$ of $\deg > 1$ then $F = x_1 H_1 + \cdots + x_n H_n \Rightarrow$ strength $\leq n$

④ If $\deg F = 2$, then $\mathrm{str}(F) = \mathrm{rank}(F)$ as a quadratic form

<u>DEF</u>: $F_1, \cdots, F_r$

collective strength $= \min \left\{ \begin{array}{l} \text{strength of non-trivial} \\ \text{homog linear comb} \end{array} \right\}$

<u>AH Principal</u>: If $F_1, \cdots, F_r$ of degree $\leq d$ have large collective strength then they behave like independent variables

<u>Ex</u>: ① $F_1, \cdots, F_r$ are algebraically independent
② $V(F_1, \cdots, F_r)$ has codim r
③ $(F_1, \cdots, F_r)$ is a prime ideal.

<u>PROP</u>: Given r, d, $\exists\, t$ s.t. for any $F_1, \cdots, F_r$ of deg $\leq d$ $\exists\, G_1, \cdots, G_t$ s.t. $F_1, \cdots, F_r$ are polynomial in G's.

[Erman - Sam - Snowden]

Idea: take strength $\to +\infty$ to get precise statement

$$R = \varprojlim \mathbb{C}[x_1, \cdots x_n]$$

A degree d element of R is a formal linear comb of deg d monomials.

<u>THM</u>: Let $\{F_i\}_{\underset{\text{uncountable}}{i \in I}}$ be a max set of ∞ collective strength. Then every element of R can be expressed uniquely as a poly in F's, i.e. $R = \mathbb{C}[T_i]_{i \in I}$

Idea of Proof:

Recall that a derivation is $\partial : A \to A$ satisfying Lebniz rule.

DEF A has enough derivations : if for any homog $r \in A$ of deg $r \geq 1$, $\exists$ homog derivation $A \to A$ of neg deg s.t. $\partial r \neq 0$

THM If A has enough derivations, then it is isomorphic to a polynomial ring.

Draisma [2017]

Let X_ℓ be the set of all homog deg d polys in ∞ vars. regard X_ℓ as an ∞-dim'l alg var.

THM [Draisma] Let Z be a GL_∞-invariant subvariety of X_ℓ (or $X_{\ell_1} \times \cdots \times X_{\ell_r}$)

Then Z is the zero locus of the GL_∞-orbits of finitely many equations.

Attempted Application:

Let $Z_n \subset X_{\ell_1} \times \cdots \times X_{\ell_n}$ be locus where pd $> n$

then $\cdots \subset Z_2 \subset Z_1$

Each Z_i is a GL_∞-loci.

Can use Draisma to show this stabilizes.