LORENTZIAN POLYNOMIALS

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A.1. **Introduction.**

$$\text{Continuous convexity} \leftrightarrow \text{Lorentzian polynomials} \leftrightarrow \text{Discrete convexity}$$

Given a polynomial $f = \sum_\alpha c_\alpha \frac{w^\alpha}{\alpha!}$ where $c_\alpha \in \mathbb{R}$ and $\frac{w^\alpha}{\alpha!} = \frac{w_1^{\alpha_1}}{\alpha_1!} \cdots \frac{w_n^{\alpha_n}}{\alpha_n!}$. We have two ways to view f. On one hand it is a function $\mathbb{R}^n \to \mathbb{R}, w \mapsto f(w)$, on the other hand, it is $\mathbb{N}^n \to \mathbb{R}, \alpha \mapsto c_\alpha$. The convexity of two different views intersects in Lorentzian polynomial.

Definition A.1. Let

H_n^d = {homogeneous degree d in n variables with $\mathbb{R}$-coefficients}

$\overset{\circ}{L}{}_n^2$ = {quadratic forms with positive coefficients that have the Lorentzian signature $(+,-,-,-)$} $\subseteq H_n^d$

$\overset{\circ}{L}{}_n^d$ = {strictly Lorentzian polynomials} = $\{f \in H_n^d \mid \partial_i f \in \overset{\circ}{L}{}_n^{d-1} \forall i\}$

The set of Lorrentzian polynomial is the closure of such $\overset{\circ}{L}{}_n^d$.

How do you check if some polynomial is Lorentzian or not? In fact, it is very easy.

Example A.2. $d = 3, n = 2, f = w_1^3 + w_2^3$. Then

$$\partial_1 f = 3w_1^2$$
$$\partial_2 f = 3w_2^2$$

We can approxiamte the first one by $3w_1^2 + 2\varepsilon w_1 w_2 + \varepsilon^3 w_2^2$, hence it's Lorentzian.

But it turns out that f is *not* Lorentzian.

Example A.3 (Newton $n = 2$). $f = \prod_{k=1}^d (w_1 + a_k w_2), a_k \geq 0$ is Lorentzian

In fact, any homogeneous stable polynomial is Lorentzian.

A.2. **Discrete Convex Analysis.**

Definition A.4. $J \subseteq \mathbb{N}^n$ is M-convex if

$$\forall \alpha, \beta \in J, \forall i \in [n], \alpha_i > \beta_i, \exists j \in [n], \alpha_j < \beta_j, \alpha - e_i + e_j \in J, \beta - e_i + e_j \in J$$

Example A.5 (Graphs). Let G be connected graphs on n edges. J = { spanning trees of G} $\subseteq \mathbb{N}^n$. Then J is M-convex

Example A.6 (Configurations of vectors). A = the set of n vectors in $\mathbb{F}^d$ where $\mathbb{F}$ is a field. J = the set of basis in $A \subseteq \mathbb{N}^n$. THen J is M-convex.

Definition A.7. Let

$M_n^d = \{f \in H_n^d \mid \text{supp}(f) \subseteq \mathbb{N}^n \text{ is M-convex}\}$ where supp is the set of all monomials in f.

L_n^2 = {quadratic forms with nonnegative coefficients that have at most one positive Lorentzian signature.}

L_n^d = {strictly Lawersain polynomials } = $\{f \in M_n^d \mid \partial_i f \in L_n^{d-1} \forall i\}$

Theorem A.8 (BH). *we have*

(1) *L_n^d is the closure of $\mathring{L}_n^d$.*
(2) *$\mathbb{P}L_n^d$ is compact contractible subset of $\mathbb{P}H_n^d$ with contractible interior.*
(3) *Let L_J be the collection of Lorentzian polynomials with support in J so that $\mathbb{P}L_n^d = \coprod_J \mathbb{P}_J$. L_J is nonempty iff J is M-convex. In this case, $\mathbb{P}L_J$ deformation retracts to the exp.gen fct $f_J = \sum_{\alpha \in J} \frac{\omega^\alpha}{\alpha!}$.*
(4) *Any nonnegative linear change of coordinates preserves L_n.*
(5) *For any homogeneous $f \in \mathbb{R}_{\geq 0}[w]$, $f \in L_n^d$ iff f is* strongly log-concave, *i.e. for any $\alpha \in \mathbb{N}^n$, either $\partial^\alpha f = 0$ or $\log(\partial^\alpha f)$ is concave in $\mathbb{R}^n_{\geq 0}$.*
(6) *Homogeneous stable polynomials are L_n^d.*
(7) *Δ_i convex body in $\mathbb{R}^d$, then the $\mathrm{vol}(w_1\Delta_1 + \cdots + w_n\Delta_n) \in L_n^d$.*
(8) *L_i nef divisors on d-dimensional projective variety, then $\deg(w_1 L_1 + \cdots + w_n L_n) \in L_n^d$.*

A.3. **Application.**

(1) Gurvits's conjecture '09 (BH'19).
(2) Okounkov's conjecture '00: $c^\nu_{\kappa\lambda}$ is log-concave in (ν, κ, λ). Chirden Derksen-weyman '07 found a counterexample

Theorem A.9 (HMMS '19)**.** *For any partition λ and any μ and any $i, j \in [n]$*

$$K_{\lambda\mu}^2 \geq K_{\lambda\mu - e_i + e_j} K_{\lambda\mu - e_j + e_i}$$

Theorem A.10 (HMMS '19)**.** *The exp.gen function $N(S_\lambda) = \sum_\mu K_{\lambda\mu} \frac{w^\mu}{\mu!}$ is Lorentzian for any λ.*

A.4. **Conjectures.** All our favourite polynomials are Lorentzian. e.g. Schubert polynomials, skew schur polynomials, skew P-poly, every homogeneous component of Gorenthendieck polynomials, key polynomials