The Hodge Conjecture

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X complex manifold. $\dim X = n$

$A^k(X) = \{\text{global } C^\infty\text{-}k\text{-forms}\} \otimes_{\mathbb{R}} \mathbb{C}$

Locally, holomorphic coordinates $(z_1, \cdots z_n)$ $z_j := (x_j, y_j)$

$$dz_j = dx_j + i\,dy_j$$

$$d\bar{z}_j = dx_j - i\,dy_j$$

$\omega = A^k(X)$ we say ω has <u>type</u> (p,q) if

locally $\omega = \sum f\, dz_{n_1} \wedge \cdots \wedge dz_{n_p} \wedge d\bar{z}_{m_1} \wedge \cdots \wedge d\bar{z}_{m_q}$

(p,q) is invariant under change of coordinates

$$\overset{(**)}{\rightsquigarrow} \quad A^k(X) = \bigoplus_{p+q=k} A^{p,q}(X) \leftarrow \text{space of } (p,q)\text{-forms}$$

Does $(**)$ descend to H_{dR}?

<u>Hodge Decomposition</u>:

If X smooth, proj, var (or compact Kahler manifold), then

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

$$H^{p,q}(X) = \frac{\text{closed } (p,q)\text{-forms}}{\text{exact } (p,q)\text{-forms}}$$

<u>Rmk</u>: $\overline{H^{p,q}(X)} = H^{q,p}(X)$

If $k = 2c$, then $\overline{H^{c,c}(X)} = H^{c,c}(X)$

The cycle class map:

X smooth complex proj variety

Poincaré Duality:

$$\int_X (-)\wedge(-) : H^k(X,\mathbb{C}) \times H^{2n-k}(X,\mathbb{C}) \to \mathbb{C}$$

is nondegenerate

$$\rightsquigarrow \quad H^k(X,\mathbb{C}) \cong H^{2n-k}(X,\mathbb{C})^\vee$$

<u>Key Observation</u>: $k = 2c$ and $p+q = 2n-k$

$$\alpha_{c,c} \wedge \alpha_{p,q} \quad \text{has type } c+p,\ c+q$$

$$= 0 \quad \text{unless } p = q = n-c$$

$$\Rightarrow \quad H^{c,c}(X) = \Big(\bigoplus_{p \neq q} H^{p,q}(X)\Big)^\perp$$

$i : S \hookrightarrow X \quad S$ smooth, $\operatorname{codim} S = c$

$$\int_S i^*(-) \in H^{2\dim S}(X,\mathbb{C})^\vee \cong H^{2c}(X,\mathbb{C})$$

so S represents an element $cl(S) \in H^{2c}(X,\mathbb{C})$

for any $\beta_{p,q} \in H^{2\dim S}(X,\mathbb{C})$

$$i^*(\beta_{p,q}) = 0 \quad \text{unless } p = q = \dim S$$

$$\Rightarrow cl(S) \in \Big(\bigoplus_{p\neq q} H^{p,q}(X)\Big)^\perp = H^{c,c}(X).$$

Remarks:

(1) $cl(S) \in H^{2c}(X, \mathbb{Z})$

(2) If $S \subseteq X$ singular irreducible, by taking a resolution of sing's $\widetilde{S} \longrightarrow S \subseteq X$, then $cl(\widetilde{S}) =: cl(S)$ still makes sense.

The Cycle Class Map:

$$cl: \underbrace{\mathbb{Z}\left\{ \begin{array}{c} \text{irreducible subsets } S \subseteq X \\ \text{of codim } c \end{array} \right\}}_{\text{algebraic cycles}} \longrightarrow \underbrace{H^{2c}(X, \mathbb{Z}) \cap H^{c,c}(X)}_{\text{Hodge classes}}$$

$\mathrm{Im}(cl)$ = algebraic classes

Hodge: cl is surjective?

Kollár: $\exists\, \alpha \in H^*(X, \mathbb{Z})$ $\alpha \notin \mathrm{Im}(cl)$, $n \cdot \alpha \in \mathrm{Im}(cl)$

Conjecture (Hodge): Every Hodge class is a rational linear combination of algebraic classes.

Ex: $X = \mathbb{P}^n$, $H^{2c}(\mathbb{P}^n, \mathbb{Z}) = \mathbb{Z}$
$cl(S) = \deg(S)$ $\Rightarrow$ cl is surj

Known Cases:

$\mathbb{P}^n$, or $\dim X = 1, 2, 3$

Unknown Cases: $\Delta \subseteq X \times X$

$$cl(\Delta) \in H^{2n}(X \times X, \mathbb{C}) = \bigoplus_{i+j=2n} H^i(X, \mathbb{C}) \otimes H^j(X, \mathbb{C})$$

$$\rightsquigarrow cl(\Delta) = \sum_{i+j=2n} cl(\Delta)_{i,j}$$ "Easy" to show $cl(\Delta)_{i,j}$ is Hodge class.

So Hodge Conjecture predicts that $cl(\Delta)_{i,j}$ is algebraic

This special implication of HC is called

"Grothendieck's standard conjecture B."

Why believe HC?

- Lefschetz (1,1) "theorem":

$$\mathrm{Pic}(X) \twoheadrightarrow H^{1,1}(X) \cap H^2(X, \mathbb{Z})$$ ← classical motivation

i.e. HC is true in codim 1.

- Cattani - Deligne - Kaplan

  Hodge classes vary algebraically in families

Variational Aspects

$\mathcal{X} \to S$ family of smooth proj variety.

$\mathcal{X}, S$ quasi-proj

⇝ holomorphic vector bundle $\mathcal{H}^{2c}$ on S

$$\mathcal{H}^{2c}_t = H^{2c}(\mathcal{X}_t, \mathbb{C})$$

2 Shocking Facts:

(1) $\mathcal{H}^{2c}$ is algebraic (Grothendieck)

total space is quasi-proj

(2) $Z = \{ (\alpha, t) \in \mathcal{H}^{2c} : \alpha \in \underbrace{H^{2c}(\mathcal{X}_t, \mathbb{Q}) \cap H^{c,c}(\mathcal{X}_t)}_{\text{expected algebraic classes}} \}$

<u>THM</u> (CDK): The connected components of Z are algebraic.