Cox rings & Mori Dream Spaces

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$\varphi : \mathbb{C}[x,y,z] \longrightarrow \mathbb{C}[t]$ $\quad a,b,c \in \mathbb{Z}^+$

$x \longmapsto t^a$

$y \longmapsto t^b$

$z \longmapsto t^c$

$P = \operatorname{Ker}\varphi$ monomial prime

$P^{(n)} = P^n R_P \cap R$ symbolic powers

$\bigoplus_{n\geq 0} P^{(n)}$ symbolic Rees algebra

In commutative alg, for what a, b, c is $\bigoplus_{n\geq 0} P^{(n)}$ Noetherian

Yes:
- $a=4$ ✓ Huneke $\quad$ 80
- $a=6$ ✓ Srinivasan $\quad$ 91
- $(a+b+c)^2 > abc$ ✓ Cutosky $\quad$ 91
- $a = m_1 b + m_2 c$ for $m_1, m_2 \in \mathbb{Z}^+$ H-K-L Recently

No: Goto - Nishida - Watanabe $\quad 7m-3,\ 8m-3,\ 5m^2-2m\ ;\ 3\nmid m, m\geq 4$

$7m-10,\ 8m-3,\ 5m^2-7m+1,\ 3\nmid m,\ m\geq 5$

Pretalk settings: Varieties over alg closed field k
- X alg var (normal)
- Weil Divisor : $\sum a_i Y_i \quad a_i \in \mathbb{Z},\ Y_i \subseteq X$ codim 1 subvar.
- Class group: $Cl(X) = WDiv(X) / PDiv(X)$

Def. $D_1, \dots, D_r$ divisors on X

The associated multigraded section ring is the graded k-alg

$$R(X; D_1, \dots, D_r) = \bigoplus_{(m_1, \dots, m_r) \in \mathbb{Z}^r} H^0(X, m_1 D_1 + \dots + m_r D_r)$$

[Hu, Keel, '00]

Def: A Mori dream space X is a normal, projective, $\mathbb{Q}$-factorial, with $\mathrm{Cl}\, X$ f.g. s.t. all multigraded section rings on X are f.g.

every codim 1 subvar Y has a multiple mY ~~that~~ is principal defined

Examples: * Toric varieties are MDS (proj. $\mathbb{Q}$-fact)

* Fano varieties ($-K_X$ ample)

Lem. Let $R = \bigoplus_{m \in \mathbb{Z}} R_m$ be a graded domain, with R_0 f.g. k-alg

For any $d \in \mathbb{Z}^+$

$$R = \bigoplus_{m \in \mathbb{Z}} R_m \text{ f.g.} / R_0 \iff R^{(d)} = \bigoplus_{m \in \mathbb{Z}} R_{md} \text{ f.g.} / R_0$$

Lem: Let $R = \bigoplus_{m \in \mathbb{Z}^n} R_m$ be a graded algebra. Let $S \subseteq \mathbb{Z}^n$ be a f.g. semigroup.

$$R = \bigoplus_{m \in \mathbb{Z}^n} R_m \text{ f.g.} / R_0 \Rightarrow R_S = \bigoplus_{m \in S} R_m \text{ f.g.} / R_0$$

<u>Def</u> Let X be a normal projective $\mathbb{Q}$-factorial var/k with f.g. class group $\mathrm{Cl}\,X$. A Cox ring of X is the k-alg

$$\mathrm{Cox}(X; D_1, \dots, D_r) = R(X; D_1, \dots, D_r)$$

where $D_1, \dots, D_r$ are generators of $\mathrm{Cl}(X)$

<u>THM</u>: X normal, proj, $\mathbb{Q}$-factorial, with $\mathrm{Cl}\,X$ f.g.

X is MDS $\iff \exists$ a $\mathrm{Cox}(X)$ f.g.

$\iff \forall\ \mathrm{Cox}(X)$ f.g.

THM (Okawa 11) [Image of MDS is MDS]

$f: X \to Y$ surj morphism of normal proj $\mathbb{Q}$-fact var

X MDS $\Rightarrow$ Y MDS.

<u>THM</u> If X & Y are isomorphic in codim 1

X MDS $\iff$ Y MDS

<u>Claim ☆</u>: $X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_N$

- Vars: Normal, proj, $\mathbb{Q}$-factorial
- Maps: surjective morphism or rat'l map isom in codim 1.

Then X_0 MDS $\Rightarrow X_N$ MDS

Hu-Keel 00: Is moduli space $\overline{M}_{0,n}$ MDS??

$\boxed{n \le 6 \text{ yes!!}}$

$n \geq 10$ NO!

$\overline{M}_{0,134} \overset{☆}{\dashrightarrow} \mathrm{Bl}_{t_0}\mathbb{P}(25, 29, 72)$

$\overline{M}_{0,13} \overset{☆}{\dashrightarrow} \mathrm{Bl}_{t_0}\mathbb{P}(7, 15, 26)$

$\overline{M}_{0,10} \overset{☆}{\dashrightarrow} \mathrm{Bl}_{t_0}\mathbb{P}(12, 13, 17)$

These are NOT MDS!

$a, b, c \in \mathbb{Z}^+$ weighted proj plane

$$\mathbb{P}(a,b,c) = \mathbb{C}^3 - \{0\} \big/ _{(x,y,z) \sim (\lambda^a x, \lambda^b y, \lambda^c z)}$$

$$Bl_{t_0} \mathbb{P}(a,b,c) \text{ is MDS} \Leftrightarrow \bigoplus_{n \geq 0} P^{(n)} \text{ is Noetherian}$$

$a, b, c \leq 200$

- only 4 tuples (a,b,c) $\underline{\underline{\text{NO}}}$ before
- got ≥ 80000 $\underline{\underline{\text{NO}}}$