# When is quasi-affine scheme affine?

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Affine $= \operatorname{Spec}(R)$

Quasi-affine $= \operatorname{Spec}(R) - V(I)$

sometimes affine, sometimes not.

When is it affine?

$X = \operatorname{Spec}(R)$

$U = \operatorname{Spec}(R) - V(I)$

when is $U \to \operatorname{Spec} \mathcal{O}(U)$ an isomorphism?

For quasiaffine schemes, pts are determined by functions vanishing on that pt.

$$\begin{aligned} U &\longrightarrow \operatorname{Spec} \mathcal{O}_X(U) \\ x &\longmapsto I_U(x) \end{aligned}$$

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, equal] & \operatorname{Spec} \mathcal{O}_X(U) \arrow[d, "\varphi"] \\ \operatorname{Spec}R - V(I) \arrow[r, hook] & \operatorname{Spec}(R) \end{tikzcd}$$

You want $\operatorname{Im}\varphi$ avoids $V(I)$

$$\varphi^{-1}(V(I)) = \phi \iff I\,\mathcal{O}(U) = \mathcal{O}(U)$$
$$\| $$
$$V(I\,\mathcal{O}(U))$$

$$\iff \Gamma_{I\mathcal{O}(U)}(-) = 0 \quad \text{on } \mathcal{O}(U)\text{-modules}$$

$$\iff H^i_{I\mathcal{O}(U)}(-) = 0 \quad \forall i \geq 0$$

## Ideals Transforms

Assume that $I$ contains a NZD on $R$.

$$\mathcal{O}(U) = \bigcup_{n=0}^{\infty} R :_{Q(R)} I^n$$

↖ total quotient ring

$$= \varinjlim_n \operatorname{Hom}_R(I^n, R)$$

$$D_I(-) = \varinjlim_n \operatorname{Hom}_R(I^n, -)$$

is the $I$-transform functor

it's left exact.

THM: $R^i D_I(M) = R^i \Gamma_I(M) \quad \forall i \geq 2$

Note $R \to D_I(R)$ is a ring hom

so $H^i_I(-) = H^i_I(D_I(-)) \cong H^i_{ID_I(R)}(-)$

$$\cong H^i_{ID_I(R)}(D_I(-)) \quad (i \geq 2)$$

Hence it's $\iff$ $H^i_I(-) = 0 \;\; \forall\, i \geq 2$ on $R$-mod

$\iff$ $D_I(-)$ is exact

$\iff$ $H^i_I(R) = 0 \;\; \forall\, i \geq 2$

$\iff$ $cd(I, R) \leq 1$

in the case $I$ contains a NZD

Then $\iff$ $cd(I, R) = 1$.