The complexity of Brauer classes

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Objects of interests

Def: F field, we say D/F is a central division alg

if $Z(D) = F$, $\dim_F D < \infty$

Ex: $\mathbb{H}/\mathbb{R}$ symbol algebras $(a,b)_{\rho,\mathbb{F}}$ ρ primitive n^{th} root of 1

given by i, j over $\mathbb{F}$

$i^n = a$, $j^n = b$, $ij = \rho ji$

they are "typically" division.

Pick $\mathbb{F} = k(t,s)$ $(t,s)_{\rho,\mathbb{F}}$

Def: An alg $A/\mathbb{F}$ is central simple if it's central & simple

A CSA (central simple alg) $\iff A \simeq M_n(D)$ central division alg (CDA)

$\iff A \otimes_{\mathbb{F}} \overline{\mathbb{F}} \cong M_n(\overline{\mathbb{F}})$ alg closure of $\mathbb{F}$

$\Downarrow$

~~Invariants~~: $\dim_{\mathbb{F}} A$ is necessarily a square

$$\deg A := \sqrt{\dim_{\mathbb{F}} A}$$

Invariants: A CSA, $A \simeq M_n(D)$

$$\operatorname{ind} A = \deg D$$

The isom classes of CDA/$\mathbb{F}$ are a group (Br : Brauer group)

$$[D_1] + [D_2] = [D_3] \quad \text{where } D_1 \otimes D_2 \cong M_n(D_3)$$

that turns out to be torsion, the order is called period.

Facts: $\mathrm{per}\, D \mid \mathrm{ind}\, D \mid (\mathrm{per}\, D)^{d_D}$

<u>Question</u>: For a given field, is there a universal bound for d_D?

↑

"period-index problem"

- $Br(\mathbb{F}_p) = 0$, $Br(\mathbb{Q}_p) = \mathbb{Q}/\mathbb{Z}$, $Br(\mathbb{R}) = \mathbb{Z}/2\mathbb{Z}$

$$0 \to Br(\mathbb{Q}) \to \bigoplus_p Br(\mathbb{Q}_p) \oplus Br(\mathbb{R}) \to \mathbb{Q}/\mathbb{Z} \to 0$$

splits noncanonically

"<u>Conjecture</u>"

For a given reasonable field $\mathbb{F}$, $\mathrm{ind}\, D \mid (\mathrm{per}\, D)^{\text{"}\dim \mathbb{F}\text{"} - 1}$

expect ~~...~~ . f.g. fields are reasonable

- alg closed field
- completion of f.type schemes.

except $\dim \mathbb{F}_p = 1$ goes up by tr. deg

$\dim \mathbb{Q} = 2$

So far

<u>THM</u> [Albert - Brauer - Hasse - Noether]

$\Rightarrow$ conj holds for $\mathbb{F}$ local or global in the sense $\operatorname{per} D = \operatorname{ind} D$

<u>M. Artin</u> [80] $\Rightarrow$ conjecture holds for $\mathbb{F} = \mathbb{C}(S)$ ← surface

if $\operatorname{per}(D) = 2^a 3^b$

then $\operatorname{per} D = \operatorname{ind} D$

Saltman [90] conj holds $\mathbb{F}$ tr.deg $1/\mathbb{Q}_p$

de Jong [Late 90] finished $\mathbb{F} = \mathbb{C}(S)$ surface

Harbater / Hartmann / Krashen $\mathbb{F} = K(X)$ X/K ~~curf~~ curve

Lieblich some conditions on K

e.g. for $\mathbb{Q}_p((t))(x)$ $\operatorname{ind} \mid \operatorname{per}^3$

For $\mathbb{Q}(X)$ no idea!

<u>THM</u> If $\mathbb{F} = \mathbb{Q}_p(S)$ where S is a surface.

then $\operatorname{ind} \mid \operatorname{per}^{?}$ if $2, p \nmid \operatorname{ind} A$

$\operatorname{ind} \mid \operatorname{per}^{?}$ if $2 \mid \operatorname{ind} A$

Methodology –

Interpret CSA's as global objects:

Endomorphisms of "twisted bundle":

Def: An Azumaya alg on X is sheaf of algs

s.t. (étale) locally $\simeq M_n(\mathcal{O}_X)$

$$\mathrm{End}(V) \sim 1$$

Given such A, choose locally $\simeq$

$$A|_U \simeq \mathrm{End}(\mathcal{O}_U^n)$$

$$\mathrm{End}(\mathcal{O}_V^n) \simeq A|_V \big|_{V \cap U}$$

get isoms $\mathrm{End}(\mathcal{O}^n_{U \cap V}) \circlearrowleft \varphi_{U,V}$

$$\downarrow$$

$$\mathcal{O}^n_{U \cap V} \circlearrowleft \tilde{\varphi}_{U,V}$$

but $\alpha_{U,V,W} = \tilde{\varphi}_{U,W}^{-1} \cdot \tilde{\varphi}_{V,W} \cdot \tilde{\varphi}_{U,V} \in GL_n(\mathcal{O}_{U \cap V \cap W})$

is trivial in $PGL_n(\mathcal{O}_{U \cap V \cap W})$

so $\alpha_{U,V,W} \in \mathbb{G}_m(U \cap V \cap W)$

$\Rightarrow$ obstruction $\in H^2(X, \mathbb{G}_m)$

so $A \rightsquigarrow \alpha \in H^2(X, \mathbb{G}_m)$

This badly glued "sheaf" is called "α-twisted sheaf"

ind(A) = min ~~deg~~ of an α-twisted coherent sheaf.
rank

Outline of argument:

$\alpha \in Br(\mathbb{Q}_p(X))$ where X scheme.

1. reduce to the case where $\underset{\substack{\| \\ \ell}}{\text{prd}}$ is prime

2. If $\ell = p$ stop

3. Choose a good model $\mathfrak{X}/\mathbb{Z}_p$ of X (after passing to ℓ covers)

4. Change the model w/ cover by ℓ^2 or ℓ^3 $(\ell = 2)$

   s.t. α extends to a global class.

$\vdots$