Local Cohomology of Thickenings.

0. Set up

$\mathbb{F}$ a field

$R = \mathbb{F}[x_1, \dots, x_n]$

I = homogeneous ideal in R

$m = (x_1, \cdots, x_n)$

$R/I^{t+1} \to R/I^t$ induces $H_m^k(R/I^{t+1})_j \to H_m^k(R/I^t)_j$ (deg j)

I. THM [BBLSZ, 2016]

char$(\mathbb{F}) = 0$, for $p \neq m$, $(R/I)_p$ a complete intersection

if $\dim \mathrm{Sing}(R/I) = i$, then for all $k < \dim(R/I) - i$,

$$H_m^k(R/I^{t+1})_j \to H_m^k(R/I^t)_j$$

is ~~event~~ eventially an isomorphism for $t \gg 0$

II. Explicit Construction for Segre product

III. "……" for determinantal case

IV. "Dao - Montaño" Thms

V. Length calculation in case of Determinantal rings.

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① $S = \mathbb{F}[a,b]/(a^2+b^2) \;\#\; \mathbb{F}[c,d]$

$= \mathbb{F}[ac, bc, ad, bd] / (\text{"}a^2+b^2\text{"})$

$= \mathbb{F}[u, v, x, y] / (u^2+v^2, x^2+y^2, ux+vy, uy-vx)$

$R = \mathbb{F}[u,v,x,y] \quad I = (u^2+v^2, x^2+y^2, ux+vy, uy-vx)$

$\dim(R/I) = 2$

THM implies $H^1_m(R/I^t)_0$

u, x generate m in R/I

$0 \to R \to R_u \oplus R_x \to R_{ux} \to 0$

in R/I, $\left(\frac{v}{u}, \frac{y}{x}\right) \mapsto \frac{vx-uy}{ux} = 0$ in R/I

define $g_1 = \frac{u^2+v^2}{u^2} = 1 + \frac{v^2}{u^2}$

$g_2 = \frac{x^2+y^2}{x^2} = 1 + \frac{y^2}{x^2}$

Claim: In general

$$\left(\frac{v}{u}\Big(\underbrace{1+\sum_{k=1}^{t-1}\frac{\binom{2k-1}{k}g_1^k}{2^{k-1}}}_{\text{first } n \text{ terms of the Taylor expension of } \frac{1}{\sqrt{1-g}}}\Big),\ \frac{y}{x}\Big(1+\sum_{k=1}^{t-1}\frac{\binom{2k-1}{k}}{2^{k-1}}g_2^k\Big)\right) \in H^1_m(R/I^t)$$

② $R = \mathbb{F}\begin{bmatrix} u & v & w \\ x & y & z \end{bmatrix}$

$$I = (\underset{\Delta_1}{\overset{\shortparallel}{vz-wy}},\ \underset{\Delta_2}{\overset{\shortparallel}{wx-uz}},\ \underset{\Delta_3}{\overset{\shortparallel}{uy-vx}})$$

$\dim(R/I) = 4$ — R/I is C-M

$\dim(\mathrm{Sing}(R/I)) = 0$ — R/I^t is NOT C-M $\forall\, t \geq 2$

[Deconci, Eisenbud & Procesi]

$H^3_m(R/I^t)_0$ is a dim 1 $\mathbb{F}$-vector space

$\ln\left(\frac{wy}{vz}\cdot\frac{uz}{wx}\cdot\frac{vx}{uy}\right) = \ln(1) = 0$

$= \ln\left(1-\left(1-\frac{wy}{vz}\right)\right)\left(1-\left(1-\frac{uz}{wx}\right)\right)\left(1-\left(1-\frac{vx}{uy}\right)\right)$

$$0 = \ln\left(1-\frac{\Delta_1}{vz}\right)\left(1-\frac{\Delta_2}{wx}\right)\left(1-\frac{\Delta_3}{uy}\right)$$

$$0 = \ln\left(1-\frac{\Delta_1}{vz}\right) + \ln\left(1-\frac{\Delta_2}{wx}\right) + \ln\left(1-\frac{\Delta_3}{uy}\right)$$

$$= \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_1}{vz}\right)^m + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_2}{wx}\right)^m + \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{\Delta_3}{uy}\right)^m$$

Use Čech complex def & this relationship to conclude that this is a nonzero element.

What if char $\mathbb{F} = p > 0$

The element persists

Ex: char $\mathbb{F} = 2$, $t = 3$

in char 0: $\Delta + \frac{1}{2}\Delta^2 \rightsquigarrow \ln(1-\Delta)$

$$\Rightarrow 2\Delta + \Delta^2 \rightsquigarrow 2\ln(1-\Delta)$$

$$\Downarrow$$

in char 2: $\Delta^2 \rightsquigarrow 0$

$t = 5$

$$\Delta + \frac{1}{2}\Delta^2 + \frac{1}{3}\Delta^3 + \frac{1}{4}\Delta^4 \rightsquigarrow \ln(1-\Delta)$$

$$\Rightarrow 4\Delta + 2\Delta^2 + \frac{4}{3}\Delta^3 + \Delta^4$$

$$\Downarrow$$

in char 2 $\Delta^4 \rightsquigarrow 0$

so there's no necessarily a limit for this process.

In char p, rank $H^3_m(R/I^t)_0 \neq 1$

as $\mathbb{F}$-vector space.

Let q = largest power of $p \leq t$

q_2 = smallest power of p such that $q + q_2 > t$

Constructed $2\lfloor \frac{q}{q_2} \rfloor - 1$ elements.

THM (Dao - Montaño, 2016)

$R = \mathbb{F}[x_1, \cdots, x_d]$, char $\mathbb{F} = 0$

I homogeneous prime s.t.

R/I is a locally complete intersection

for every $k < \dim(R/I) - \dim(\operatorname{sing}(R/I))$

such that $H^{d-k}_I(R) \neq 0$

we have $\underline{\lim}_{t} \dfrac{\lambda(H^k_m(R/I^t))}{t^d} > 0$

THM (Dao and Montaño, 2016)

$R = \mathbb{C}[X_{ij}]$, where X an $m \times n$ matrix

I - GL-invariant ideal

that is the thickening of a determinantal ideal

then for every $k \leq m+n-2$,

we have $\overline{\lim}_{t \to \infty} \dfrac{\lambda(H^k_m(R/I^t))}{t^d} < \infty$

$R = \begin{bmatrix} x_{11} & \cdots & x_{1m} \\ x_{21} & \cdots & x_{2m} \end{bmatrix}$, I gen by 2×2 minors

$$\lim_{t\to\infty} \frac{\lambda(H^3_m(R/I^t))}{t^{2m}} = \frac{1}{(m+1)(m!)^2}$$

then $d! *$ ↑ gives the m^{th} Catalan number.