Introduction to the MMP

Goal: Classify smooth, proj. complex. alg varieties

↳ up to birational equivalence

I. Surfaces

Recall: Let X be a projective variety

$C \subset X$ smooth curve

D a Cartier divisor on X

$$(C \cdot D) = \deg \mathcal{O}_X(D)|_C$$

i.e.

Def ~~Def~~ Let X be a smooth proj ~~curve~~ surface

A curve $E \subset X$ is ~~closed~~ called (-1)-curve

if $\cdot\ E \sim \mathbb{P}^1$

$\cdot\ E^2 = E \cdot E = -1$

Ex: If X is a smooth surface and $p \in X$

then $\mathrm{Bl}_p X \to X$ E is a (-1)-curve.

$$\begin{array}{c} \mathrm{Bl}_p X \\ \cup \\ E \end{array}$$

Theorem (Castelnuovo)

Let X be a smooth projective surface, and $E \subset X$ a (-1)-curve.

Then there is a smooth proj variety Y, $p \in Y$

and $f : X \to Y$

s.t.
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, phantom, "\cong" rotate=90] & Y \\ Bl_p Y \arrow[ur] & \end{tikzcd}$$
with exceptional divisor E.

f is called blowing down.

Remark: Set $\rho(X) = \dim_{\mathbb{Q}} NS(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

(1) Sever's theorem: $\rho(X) < \infty$

(2) $\rho(Bl_p Y) = \rho(Y) + 1$

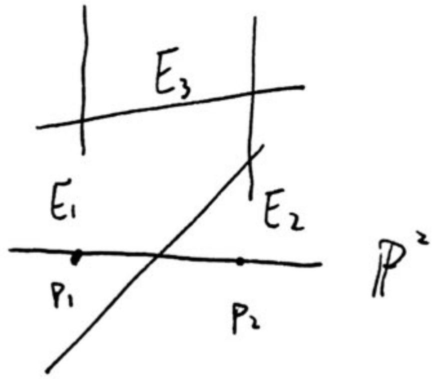
$\Rightarrow$ blow-down process terminates

THM: If $f : X \to Y$ is a birational morphism of smooth proj surfaces

then f is a finite seq of blow-ups at points.

Cor: If X has no (-1)-curve, then there's no ~~nonrational~~ nontrivial rational morphism out of X.

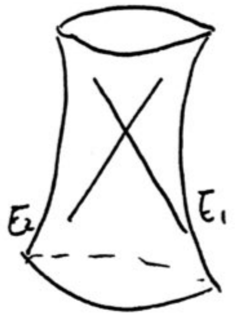
Ex



blow up twice $Bl_{p_1,p_2}\mathbb{P}^2$

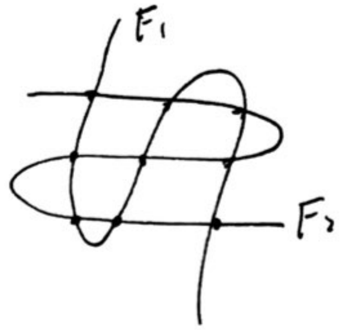
get 3 (−1)-curves E_1, E_2, E_3

blow down E_3 →



Now 茜 it has no (−1)-curve.

Ex

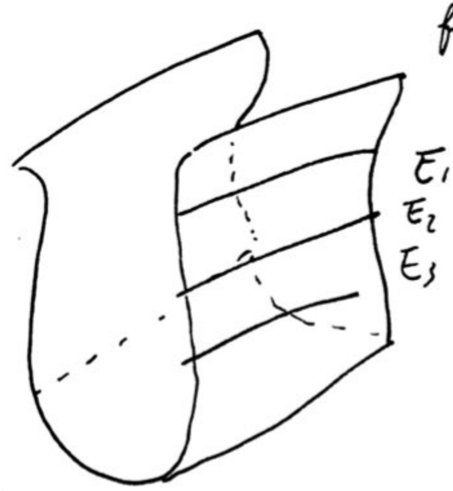


birational morphism

→ $[F_1 : F_2]$

honest morphism with elliptic curve fiber

blow-up at 9 pts



at least $\mathbb{Z}^8$ (−1)-curves by group structure for each elliptic curve.

Higher dimension:

Def. X a smooth proj variety

define $N_1(X) = \dfrac{\mathbb{R}\{C : C \subset X \text{ a curve}\}}{C_1 \sim C_2 \text{ if } C_1 \cdot D = C_2 \cdot D \ \forall D \text{ divisor}}$

by definition, ~~choose a perfecting~~ perfect pairing

$$N_1(X) \otimes NS(X)_{\mathbb{R}} \to \mathbb{R}$$

$$\Rightarrow N_1(X) \text{ finite dim}$$

Define $\overline{NE(X)} = \overline{\mathbb{R}_{\geq 0} \langle C : C \text{ effective} \rangle}$

the cone of curves of X

Rmk: If $g : X \to Y \rightsquigarrow g_{NE} : \overline{NE(X)} \to \overline{NE(Y)}$

Check: If curve $C \subset X$, then $g(C) = pt$ iff $g_*([C]) = 0$

$\hookrightarrow$ being contracted is a "numeric" property of a curve.

Def If X is a normal proj variety

$F \subset \overline{NE(X)}$ an extremal face of $\overline{NE(X)}$

then the/a contraction morphism ~~with~~ out X is a proj mor

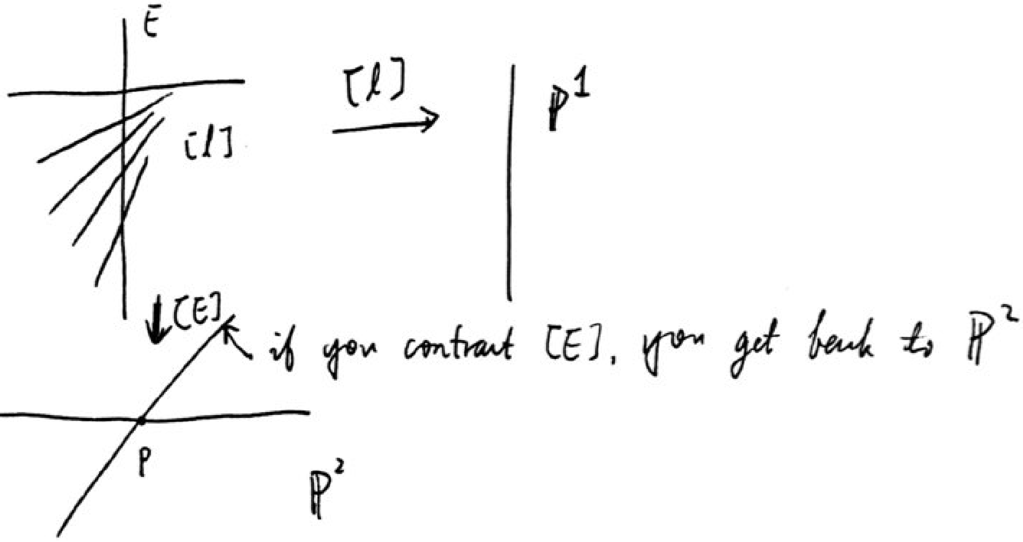
$f : X \to Y$, Y normal

(1) $g_* \mathcal{O}_X = \mathcal{O}_Y$

(2) a curve C is ~~contract~~ contracted by g iff $[C] \in F$

Converse: For any morphism $f: X \to Y$

we get a extremal face of $\overline{NE(X)}$



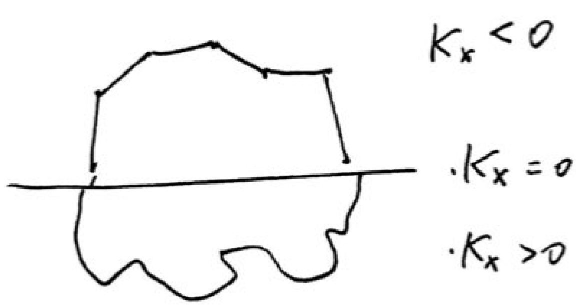
Thm Let X be a smooth proj var.

(1) (Cone) $\exists$ ~~Countably~~ Countably many rational curves $C_i \subset X$

s.t. $-2 \dim X \leq K_X \cdot C_i < 0$

these C_i

s.t. $\overline{NE(X)} = NE(X)_{\cdot K_X > 0} + \sum \mathbb{R}_{\geq 0} [C_i]$



(2) (Contraction thm) For each $[C_i]$, the associated contraction morphism exists

Ex Let A be a 3-dim abelian variety

Let $i: A \to A \quad x \mapsto -x$

There are $64 = 2^{3\cdot 2}$ fixed points

Let $\pi: A \to X = A/i$

around fixed pts, locally looks like $\operatorname{Spec} \mathbb{C}[x^2, y^2, z^2, xy, xz, yz]$

Let $f: Y = Bl_{64 \text{ pts}} X$, then Y is smooth

One can show that Y has 64 ~~curves~~ E_i s.t. $E_i \cdot K_X < 0$
exceptional div

so blow down Y at these 64 E_i gives an A/i, i.e. with 64 singular pts.