Positivity of Divisors

X = smooth proj var / $\mathbb{C}$

<u>Idea</u> study X via the "positive divisors" that it admits

<u>Notation</u>

$\mathrm{Div}(X)$ = group of divisors

$$N^1(X) = \mathrm{Div}(X) / \equiv_{num}$$

where $D \equiv_{num} D'$ if $\#(D \cdot C) = \#(D' \cdot C)$

$\forall\, C \subseteq X$ curve irreducible

$$N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R}$$

Néron-Serai space

<u>Positivity Notation</u> #1 : Ampleness

Recall: $D \in \mathrm{Div}(X)$ <u>ample</u>

if $X \xrightarrow{\phi_{|nD|}} \mathbb{P}^N$ for $n, N > 0$

Equivalently, intersection theoretic criterion

$$D = \text{ample} \iff \forall\, V \subset X, \ (D^{\dim V} \cdot V) > 0$$

irred subvariety

$\rightsquigarrow$ a "limit" of ample divisors:

$$\forall\, V \subset X, \ (D^{\dim V} \cdot V) \geq 0$$

Def D is nef if the above condition holds

~~D is nef~~

Equivalently (Kleiman):

D is nef iff $(D \cdot C) \geq 0$

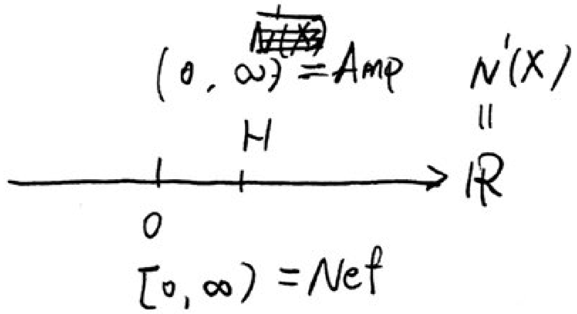
$\forall\, C \subseteq X$ curve

$Amp(X) \subseteq N^1(X)_{\mathbb{R}}$ open cone spanned by ample divisors

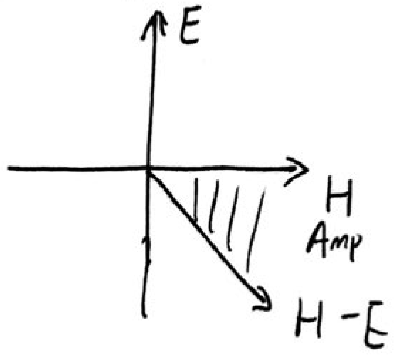
$Nef(X) \subseteq N^1(X)_{\mathbb{R}}$ closed cone spanned by nef divisor

Ex 0 $X = \mathbb{P}^2$, $Pic\, X = \mathbb{Z} \cdot H$

$N^1(X) = \mathbb{Z} \cdot H$



Ex 1 $X = Bl_p \mathbb{P}^2$ $E \subseteq X$ exceptional divisor

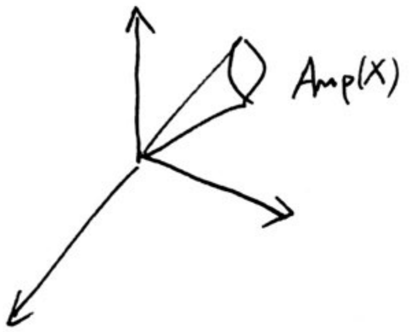


$N^1(X)_{\mathbb{R}} = \mathbb{R}^2$

<u>Ex 2</u> $X = E \times E$ where E = elliptic curve

$N^1(X)_{\mathbb{R}} = \mathbb{R}\langle f_1, f_2, \delta \rangle$ f_i : i^{th} fiber

δ : diagonal



Natural Q's:

(a) $\mathrm{int}(\mathrm{Nef}(X))$? $\mathrm{Amp}(X)$

(b) $\overline{\mathrm{Amp}(X)}$? $\mathrm{Nef}(X)$

(c) $\mathrm{Nef}(X)^\vee$?

$$N_1(X) = \mathbb{Z}\{\text{irred curves}\} / \equiv_{num}$$

$$N_1(X)_{\mathbb{R}} = N_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$$

$$\mathrm{Nef}(X)^\vee = \{ C \in N_1(X)_{\mathbb{R}} : (D \cdot C) \geq 0 \ \forall D \in \mathrm{Nef}(X) \}$$

$$= \overline{NE}(X)$$

closure of cone of positive int comb's of irred curves

Positivity Notion #2: Effectivity

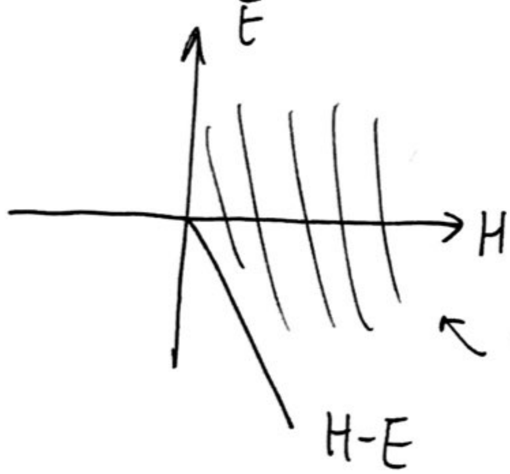
<u>Recall</u>: $D \in \mathrm{Div}(X)$ is <u>effective</u> if it's a $\mathbb{Z}_{\geq 0}$-linear comb of irreducible divisors.

$\rightsquigarrow \underset{\text{effective cone}}{\mathrm{Eff}(X)} \subseteq N^1(X)_{\mathbb{R}}$

<u>Goal</u>: (1) Replicate same story as for ampleness

(2) concrete generic info about X

<u>Ex 1</u> $\quad X = Bl_p \mathbb{P}^2$

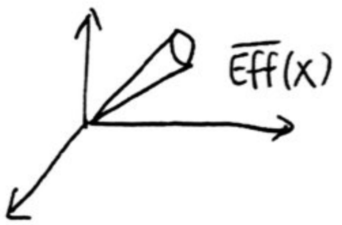


$N^1(X)_{\mathbb{R}} = \mathbb{R}^2$

<u>Def</u> Pseudoeffective cone (psef) $= \overline{\mathrm{Eff}}(X) \subseteq N^1(X)_{\mathbb{R}}$

<u>Ex 2</u> $\quad X = E \times E \quad E$ elliptic curve

$N^1(X)_{\mathbb{R}} = \mathbb{R}^3$



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<u>Ex 3</u> $X = Bl_{16}\mathbb{P}^2$

$$D = 4H - \sum_{i=1}^{16} E_i \quad \cancel{= -\omega_X}$$

Claim: D is psef but not effective

Why not effective?

$H^0(nD) \longleftrightarrow$ homogeneous polys in $\mathbb{P}^2$ of deg $4n$ vanish with multiplicities n at 16 points.

$$\begin{pmatrix}\text{expected}\\ \text{dim}\end{pmatrix} = \binom{2+4n}{2} - \sum_{1}^{16}\binom{(n-1)+2}{2}$$

$= 1-2n \quad < 0$ if n is large, i.e. $n \geq 1$

so it doesn't have any global sections

<u>Q</u>: (a) $\mathrm{int}(\overline{\mathrm{Eff}}(X)) = ?$

(b) $\overline{\mathrm{Eff}}(X)^\vee = ?$

For (a): candidate: (effective) + (very positive)

<u>Def</u> D is big if $\exists m$, $mD \equiv_{num} E + A$
(E eff, A ample)

$Big(X) \subseteq N^1(X)_{\mathbb{R}}$ big cone.

<u>THM</u> Define $\mathrm{vol}: N^1(X)_{\mathbb{R}} \to \mathbb{R}$

$$D \longmapsto \varlimsup_{m\to\infty} \frac{h^0(mD)}{m^{\dim X}/\dim X!}$$

Then $Big(X) = \{\mathrm{vol} > 0\}$

$$\Rightarrow \begin{cases} Amp(X) \subseteq Big(X) \subseteq \overline{\mathrm{Eff}}(X) \\ Nef(X) \subseteq \overline{\mathrm{Eff}}(X) \end{cases}$$

<u>Ex</u>: Big $\neq$ Amp

$X = Bl_p \mathbb{A}^2$

For $\ell \geq 2$, $D_\ell = \underbrace{(\ell-2)E}_{\text{eff}} + \underbrace{(H - \frac{1}{2}E)}_{\text{ample}}$

but D_ℓ is not ample

$(D_\ell \cdot D_\ell) = 1 - \ell^2 < 0$

For (b): candidates

Curve $C \subseteq X$ s.t.

$\forall E \subsetneq X$, can move

C s.t. $C \not\subseteq E$

intersect $\dim X - 1$

ample div's

"movable curves"

$\overline{Mov}(X)$

"Strongly movable curves"

$\overline{SMov}(X)$

THM [BDPP, '04] $\overline{Eff}(X)^\vee = \overline{Mov}(X)$

$= \overline{SMov}(X)$

<u>Cor</u> $K_X \notin \overline{Eff}(X) \iff X$ = uniruled

i.e. $\exists$ dom rat'l map

$Y \times \mathbb{P}^1 \dashrightarrow X$

<u>Pf</u>: $(\Rightarrow)$ BDPP $\Rightarrow \exists$ movable curve C_t s.t. $(K_X \cdot C_t) < 0$

Miyaoka-Mori (86): $\exists \mathbb{P}^1$ through any pt of C_t

$$\therefore \bigcup_t C_t \times \mathbb{P}^1 \dashrightarrow X$$

$(\Leftarrow)$ WTS $K_X \notin Big(X)$

will show $H^0(nK_X) = 0 \quad \forall n > 0$

if $\exists$ nonzero $\omega \in H^0(nK_X)$

$$\phi^* \omega \in H^0(nK_{Y \times \mathbb{P}^1})$$

$$= H^0(nK_Y) \otimes \underbrace{H^0(nK_{\mathbb{P}^1})}_{0} \quad \cdots \quad \#$$

<u>Conj</u> (Weak) [Abundance conj]

X = uniruled $\Leftrightarrow H^0(nK_X) = 0 \quad \forall n > 0$