Associated Primes & the Local Cohomology of Parameter Ideals

(R, m, K) complete local ring containing a field

<u>THM</u> [1993, Huneke & Sharp (char p), Lyubeznik (char 0)]

If R is regular, then for any ideal $I \subseteq R$

$$\mathrm{Ass}_R H^i_I(R) \text{ is finite } \forall i \geq 0$$

<u>Ex</u>: $R = \dfrac{K[[\begin{smallmatrix} s & u & x \\ t & v & y \end{smallmatrix}]]}{su^2x^2 - (s+t)uvxy + tu^2y^2}$

(Katzman 2002) $\mathrm{Ass}_R H^2_{(u,v)}(R)$ is ∞!

<u>THM</u> [2017 Hochster & Núñez-Betancourt; Katzman & Zhang]

If R is a hypersurface in char $p > 0$, then $\mathrm{supp}_R H^i_I(R)$ is Zariski <u>closed</u> (equivalently, $\mathrm{Min}_R H^i_I(R)$ is finite)

Q: If R is a complete intersection, is $\mathrm{supp}\, H^i_I(R)$ closed?

Notation:
- (B, m_B, K) is a complete RLR containing a field
- $J = (\underbrace{f_1, \cdots, f_m}_{\text{partial s.o.p of } B})$
- $I \subseteq B$ any ideal
- $R = B/J$

H & NB : If R has char p.

then $\mathrm{Ass}_B H_I^{i+1}(J)$ finite $\Rightarrow \mathrm{supp}_R H_I^i(R)$ is closed

Q: If $J \subseteq B$ is a parameter ideal, is $\mathrm{Ass}_B H_I^i(J)$ always finite?

A: No

$i=0$: $H_I^0(J) = 0$ ✓

$i=1$: THM [2000, Bradmann & hashyam - Fachani] ✓

If A is Noetherian, $I \subseteq A$ an ideal

M is f.g. A-module and

$H_I^i(M)$ is f.g. $\forall j < i$

then $\mathrm{Ass}_A H_I^j(M)$ is finite

Our goal:

$i=2$: Works even if J is not a parameter ideal

$i=3$ & 4: works if J is generated by part of a r.s.o.p

However, if J is not generated by r.s.o.p's

then $\mathrm{Ass}_R H_I^3(J)$ can be ∞.

Finiteness of H^2: (B is a UFD)

Apply $\Gamma_I(-)$ to $0 \to J \to B \to B/J \to 0$

$$0 \to 0 \to 0 \to \Gamma_I(B/J) \to H^1_I(J) \to H^1_I(B) \to H^1_J(B/J) \to H^2_I(J) \to H^2_I(B) \to H^2_I(B/J) \to \cdots$$

if $\operatorname{depth}_I(B) \geq 2$:

$$0 \to H^1_J(B/J) \to H^2_I(J) \to N \to 0, \quad N \subset H^2_I(B)$$

B & LF $\Rightarrow$ finite Ass ... ① B & LF $\Rightarrow$ finite Ass

$\Rightarrow H^2_I(J)$ has finite Ass

Real work: $\operatorname{depth}_I(B) = 1$

R is a UFD (up to radical) write $I = rB \cap I_0$

$\operatorname{depth}_{I_0} B \geq 2$

Idea Transforms:

$$D_I(-) := \varinjlim_t \operatorname{Hom}_B(I^t, -)$$

key facts:

⓪ left exact, covariant

① $R^i D_I(-) = H^{i+1}_I(-)$ $\underset{\wedge}{\overset{i\geq 1}{}}$ (Apply $\operatorname{Hom}_B(-, M)$ to $0 \to I^t \to B \to B/I^t \to 0$)

② For any B-module M.

$$0 \to H^0_I(M) \to M \to D_I(M) \to H^1_I(M) \to 0 \text{ is exact.}$$

③ $H^0_I(D_I(M)) = H^1_I(D_I(M)) = 0$

$\forall i \geq 2$: $H^i_I(D_I(M)) \cong H^i_I(M)$

④ $I = rB$, $D_I(-) = (-)_r$ ← localization

<u>Prop</u>: If $I = rB \cap I_0$
then $D_I(-) = (D_{I_0}(-))_r$

<u>COR</u>: $\forall i \geq 2$, $H^i_I(-) \simeq (H^i_{I_0}(-))_r$

It suffices to show that $\mathrm{Ass}_B H^2_{I_0}(J)$ is finite
but now $\mathrm{depth}_{I_0}(B) \geq 2$ done
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<u>THM</u> [-] If J is a regular s.o.p in a complete RLR B, $I \supseteq J$
then $\mathrm{Ass}_B H^3_I(J)$ & $\mathrm{Ass}_B H^4_I(J)$ are finite

<u>LEM</u> If $\mathfrak{a} \supseteq J$ such that $\mathrm{Ass}_B H^i_{\mathfrak{a}}(J)$ is ∞
then there exists $\mathfrak{U} \supseteq \mathfrak{a}$ such that
$\mathrm{Ass}_B H^i_{\mathfrak{U}}(J)$ is finite &
$d = \mathrm{depth}_{\mathfrak{U}}(R)$ satisfies $i = d+2$ or $i = d+(M+1)$

Ex $A = K[\![\begin{smallmatrix} s & u & x \\ t & v & y \end{smallmatrix}]\!]$ $\qquad R = A/f$

$$f = su^2x^2 - (s+t)vxuy + tu^2y^2$$

then Katzman : $H^2_{(u,v)}(A/f)$ has ∞ Ass_A

Key idea: present A/f as a C.I. in higher codimension

$$B = A[\![w]\!] \,,\; J = (w, f) \,,\; B/J \cong R$$

$$I = (w, f, u, v) \qquad IR = (u,v)R$$

$\operatorname{depth}_I B = 3$ $\qquad w, f, u$ is a reg seq

Apply $\Gamma_I(-)$ to $0 \to J \to B \to R \to 0$

$$0 \to 0 \longrightarrow 0 \longrightarrow 0 \to$$

$$\to H^1_I(J) \longrightarrow 0 \to H^1_I(R) \to$$

$$\to H^2_I(J) \to 0 \to H^2_I(R) \leftarrow \infty \text{ Ass}$$

$$\to H^3_I(J) \to H^3_I(B) \to \cdots$$

$$\Uparrow$$

$$\infty \text{ Ass}$$