Hilbert Multiplicities

(R, m, K) local (Noetherian) ring

<u>Note</u>: If $I \subseteq R$ is an ideal, then $\sqrt{I} = m \Leftrightarrow \dim_K (R/I) < \infty$

<u>Ex</u>: $R = K[[x, y]]$ $\quad I = (x^2, xy, y^2)$

$$\ell(R/I) = 3 \qquad \sqrt{I} = (x, y)$$

From now on, all ideals I in this talk will be $(\sqrt{I} = m)$ m-primary

The Hilbert-Samuel function

$$HS_M^I(n) = \ell(M/I^nM) \text{ for } M \text{ a f.g. } R\text{-module.}$$

THM: For n sufficiently large. $HS_M^I(n)$ is a polynomial of deg $\dim(M)$ in n.

Sadly $HS_M^I(n)$ is not additive on exact sequences.

$$0 \to M' \to M \to M'' \to 0$$

Notice

$$0 \to \boxed{\frac{M' \cap I^nM}{I^nM'}} \to M'/I^nM' \to M/I^nM \to M''/I^nM'' \to 0$$

so $HS_{M'}^I(n) + HS_{M''}^I(n) = HS_M^I(n) + HS_{\square}^I(n)$

<u>Artin-Rees lemma</u> If $M' \subseteq M$, ~~then~~ $I \subset R$ an ideal, then there exists an $k \geq 1$ s.t. $\forall n > k$

$$I^n M \cap M' = I^{n-k}(I^k M \cap M')$$

Goal: Show $l\left(\frac{M' \cap I^n M}{I^n M'}\right)$ has deg $< d$

$$l\left(\frac{M' \cap I^n M}{I^n M'}\right) \leq l\left(\frac{I^{n-k} M'}{I^n M'}\right)$$

Use
$$0 \to \frac{I^{n-k} M'}{I^n M'} \to M'/I^n M' \to M'/I^{n-k} M' \to 0$$

$$\Rightarrow \qquad = l\left(M'/I^n M'\right) - l\left(M'/I^{n-k} M'\right)$$

$$HS^{I}_{\square}(n) = \underbrace{HS^{I}_{M'}(n) - HS^{I}_{M'}(n-k)}_{\text{top degree cancels}}$$

<u>Def</u> The Hilbert-Samuel multiplicities of M on I is

$$e_I(M) = \lim_{n \to \infty} \frac{d!\, l(M/I^n M)}{n^d} \qquad d = \dim(M)$$

<u>Ex</u>: $R = K[x,y]$ $\quad I = (x,y)$

$HS(1) = 1 \quad HS(2) = 3 \quad HS(3) = 6$

$$\ldots HS(n) = \binom{n+1}{2} = \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$e_I(R) = 1$$

<u>Ex</u>: $R = k[x_1, \cdots, x_d]$

$$HS(n) = \binom{n+d-1}{d} \Rightarrow e_I(R) = 1$$

THM: If R is regular, then $e_m(R) = 1$

<u>Ex</u>: $R = \dfrac{k[x,y]}{(x^2, xy)}$ $\quad \ell\left(\dfrac{R}{m^n}\right) = n+1$

$$\Rightarrow e_m(R) = 1$$

the THM is iff if we restrict to unmixed case.

$\updownarrow$

$\dim(R/p)$ is constant for all $p \in Ass(R)$

$\uparrow$

$Ass((0))$