Refs

Char 0: [JM] • Jonsson-Mustaţă

"Valuation and Asymptotic Invariants & Sequences of Ideals"

• Boucksom - de Fernex - Favre - Urbinati

"Valuation Spaces and Multiplier Ideals on singular varieties"

Char $p>0$: Canton

"Berkovich log Discrepancies in positive char"

Intro to Frigid Geometry

X = ~~elle~~ excellent int scheme over a field k

$L = \mathrm{Frac}(X)$

Def: ① A $\mathbb{R}$-valuation v on L is centered on X if $\exists\, x \in X$ s.t. $\mathcal{O}_{X,x} \subseteq A_v = v^{-1}[0,+\infty]$

② If v is centered on X, $\exists!\, x \in X$ s.t. $\mathcal{O}_{X,x} \subseteq A_v$ is local

$(m_v \cap \mathcal{O}_{X,x} = m_x)$

③ This unique $x \in X$ is called the center of $\underbrace{v \text{ on } X}_{c_X(v)}$

④ $\mathrm{Val}_X = \{$ valuations v on L centered on X

- If X is a projective variety/k (proper scheme/k) then every valuation on L is centered on X (valuation criterion for properness)
- $X = \mathrm{Spec}(k[t]) = \mathbb{A}^1$ then $\mathrm{ord}_{(t^{-1})} \notin \mathrm{Val}_{\mathbb{A}^1}$

Examples of $v \in \mathrm{Val}_X$

⓪ Say X is normal (really R_1) and $E \subset X$ prime divisor

$\mathcal{O}_{X,E}$ is a DVR $\Leftrightarrow$ $\mathrm{ord}_E : L^* \to \mathbb{Z}$

so $\mathrm{ord}_E \in \mathrm{Val}_X$

For $c \in (0,\infty) \subseteq \mathbb{R}$, $(c\cdot\mathrm{ord}_E : L^* \to c\cdot\mathbb{Z}) \in \mathrm{Val}_X$

① $\pi : Y \to X$ birational from normal scheme Y

$\forall E \subset Y$ prime divisor, $c\cdot\mathrm{ord}_E \in \mathrm{Val}_X$
$c>0$

$\overline{\pi(E)} \subset X$ is a irreducible closed subset

If η = generic point of $\overline{\pi(E)}$

then $C_X(c\cdot\mathrm{ord}_E) = \eta \quad \forall c>0$

② $\mathrm{triv}_X : L^\times \to \{0\}$

③ Suppose $R = K[[x_1, \dots, x_d]]$

Pick $\vec{r} \in (\mathbb{R}_{\geq 0})^d$ $\quad \vec{r} = (r_1, \dots, r_d) \quad r_i \geq 0$

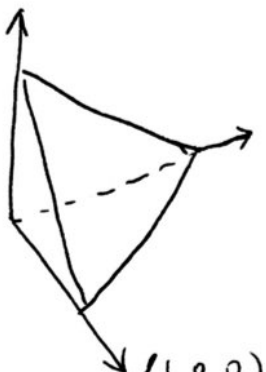
$$\mathrm{val}_{\vec{r}}(f) = \min \left\{ \sum_{i=1}^{d} r_i \alpha_i \;\middle|\; c_\alpha \neq 0 \right\}$$

where $f = \sum_{\alpha \in \mathbb{N}^d} c_\alpha \underline{x}^\alpha$, $c_\alpha \in K$

[JM] Prop 3.1: $\mathrm{val}_{\vec{r}}$ is well-defined

Moreover, $\mathrm{val} : (\mathbb{R}_{\geq 0})^d \hookrightarrow \mathrm{Val}_{\mathrm{Spec}(R)}$

$(\mathbb{R}_{\geq 0})^3$



$(1,0,0) \rightsquigarrow \mathrm{ord}_{(1,0,0)} = \mathrm{ord}_{x_1}$

④ Suppose $\pi : Y \to X$ birational from regular scheme Y

and $H = \sum_{i=1}^{n} H_i$ is an SNC divisor on Y

with each $H_i \subset Y$ regular irred hypersurface

For $\bigcap_{i=1}^{m} H_i = Z_1 \cup \cdots \cup Z_t$

$(m \leq n)$ irred decomposition

Let η = generic point of Z.

$\mathcal{O}_{Y,\eta}$ is a RLR of dim d

$\hat{\mathcal{O}}_{Y,\eta} \cong K(\eta)[[z_1, \cdots, z_d]]$

where $H_i \cap \operatorname{Spec}(\mathcal{O}_{Y,\eta}) = \mathbb{V}(z_i)$

↑ because these are SNC.

Now we have

$\mathrm{val}_{H}: (\mathbb{R}_{\geq 0})^d \longrightarrow \mathrm{Val}\, \hat{\mathcal{O}}_{Y,\eta}$

s.t. $\vec{r} \longmapsto \mathrm{val}_{H,\vec{r}} = \min\{\vec{r}.\alpha \mid C_\alpha \neq 0\}$

actually can be restricted

$\mathcal{O}_{X,\pi(\eta)} \hookrightarrow \mathcal{O}_{Y,\eta} \hookrightarrow \hat{\mathcal{O}}_{Y,\eta}$ to give $\mathrm{val}_{H,\vec{r}} \in \mathrm{Val}_X$

where $C_X(\mathrm{val}_{H,\vec{r}}) = \pi(\eta)$ $(\ast)$

so we get $\mathrm{Im}(\mathrm{val}_H) \subset \mathrm{Val}_X$

- denote this by $QM_\eta(Y, H)$
- call these <u>quasi-monomial valuations</u>

(5) $R = k[x_1, \cdots, x_d]$

$A = k[[t]]$

$\mathrm{Frac}(A) = k((t))$ has ∞-transcendental deg over k

(Ex: Show $\mathrm{trdeg}(K((t))) = \mathrm{Card}(P(\mathbb{N}))$)

In any case, choose $d-1$ transcendental series $f_2, \cdots, f_d \in k[[t]]$ that are alg indep w/o constant terms

$\varphi: R \to k[[t]]$

$\varphi(x_1) = t, \quad \varphi(x_i) = f_i \quad i \geq 2$

This is injection define $v = \mathrm{ord}_E \circ \varphi$ is a valuation on

Spec (R), centered at $(x_1, \cdots, x_d)$

* Next time: this doesn't come from ord_E for any

$\pi: Y \to \mathbb{A}^d$ (This is not Abhyankar)

Topology on Val_X: $L = \mathrm{Frac}(X)$

3 equivalent ways (product topology)

① $\mathrm{Val}_X \hookrightarrow \prod_{f \in L^\times} (-\infty, +\infty)$

$v \longmapsto \prod v(f)$

give Val_X the subspace topology

② Each $f \in L^\times$ gives a function $\hat{f}: \mathrm{Val}_X \to \mathbb{R}$

$v \mapsto v(f)$

"Gelfand transformation of f"

Give Val_X the coarsest topology s.t. $\hat{f}$ is continuous $\forall f \in L^\times$

③ Fix $v \in \mathrm{Val}_X$. A basis for the topology near v is given as follows:

Choose $f_1, \cdots, f_s \in L^\times$, $\varepsilon > 0$

$$\mathcal{U} = \bigcap_{i=1}^{s} \{ w \in \mathrm{Val}_X : |\hat{f}_i(w) - \hat{f}_i(v)| < \varepsilon \}$$

<u>Facts</u> : • Val_X is Hausdorff

• $c_X : \mathrm{Val}_X \to X$
is anticontinuous

c_X^{-1}(closed) is open