Syzygies of Homogeneous Coordinate rings

Alex Küronya

<u>Aim</u>: Study free resolutions of graded rings from alg. geom.

(work / $\mathbb{C}$)

- If $X \xhookrightarrow{i} \mathbb{P}$ the coordinate ring $R(X;i)$ is a graded ring & conversely if known $R(X,i)$ then it determines an injection $X \hookrightarrow \mathbb{P}$

More abstractly, if X = variety & $\mathcal{L}$ sufficiently positive line bundle, then the global sections yields $X \dashrightarrow \mathbb{P}$

$X \hookrightarrow \mathbb{P}$

$$(X, \mathcal{L}) \approx X \hookrightarrow \mathbb{P}$$

(~~Tacti~~ Tacitly, assume $\mathcal{L}$ very ample)

<u>Idea</u>: $R(X,i)$ becomes a graded module $/ S = \mathbb{C}[x_1, \dots, x_n]$

$\rightsquigarrow$ admits a minimal graded free resolution

$E_\bullet \to R(X,i)$ want $E_\bullet$ to be as simple as possible.

$$\begin{array}{c} 0 \\ \uparrow \\ R(X,i) \\ \uparrow \\ E_0 = S \oplus \bigoplus_j S(-a_{0j}) \\ \uparrow \\ E_1 = \bigoplus_j S(-a_{1j}) \\ \uparrow \\ \vdots \end{array}$$

Min: $a_{ij} \geq i+1$

This leads to

Def. (Green - Lazarsfield)

$R(X,i)$ (the pair (X,L)) satisfies property (N_p) $(p \in \mathbb{N})$

if $E_0 = S$ & $a_{ij} = i+1$ $\forall\, 1 \le i \le p$, & $\forall j$

In geometric terms:

$(N_0) \iff X \underset{\substack{\text{proj} \\ \text{normal}}}{\hookrightarrow} \mathbb{P}$

$(N_1) \iff (N_0)$ & the $I_{X/\mathbb{P}}$ is generated by quadratics

$(N_2) \iff (N_1)$ & the relation among generating quadratics generated linear elements.

Example: A rational normal curve $C \hookrightarrow \mathbb{P}^3$

This satisfies (N_2)

$I_{C/\mathbb{P}^3} = (y^2 - xz,\ yz - xw,\ z^2 - yw) \subset \mathbb{C}[x,y,z,w]$

The minimal free resolution of $R(C,i)$ is

$$0 \to S(-3)^2 \xrightarrow{\begin{pmatrix} w & -z & y \\ z & -y & x \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} y^2 - xz \\ yz - xw \\ z^2 - yw \end{pmatrix}} S \longrightarrow R \to 0$$

More generally, the irrational normal curve $C \hookrightarrow \mathbb{P}^n$ satisfies prop N_{n-1}

Ex: C elliptic curve

If $\mathcal{L}$ - line bundle of deg ≥ 3

then $i_{\mathcal{L}} : C \hookrightarrow \mathbb{P}$

deg $\mathcal{L} = 3$: $C \hookrightarrow \mathbb{P}^2$ as a smooth cubic curve.

$R(C, i_{\mathcal{L}})$ is normal $\Rightarrow R(C, i)$ satisfies (N_0)

but $I_{C/\mathbb{P}^2}$ is generated by a cubic

$\Rightarrow R(C, i_{\mathcal{L}})$ is not (N_1)

General picture: if deg $\mathcal{L} = p+3$, then $R(C, i_{\mathcal{L}})$ satisfies (N_p)

but not (N_{p+1})

Thm (Green 1984) Let $p \in \mathbb{N}$, C= smooth proj curve of genus g

nondegenerate $C \hookrightarrow \mathbb{P}^n$ with $n \geq g+p+1$

Then $R(C, \hookrightarrow)$ satisfies (N_p)

(If $\mathcal{L}$ s.t. $\deg_C \mathcal{L} \geq 2g+p+1$, then $\mathcal{L} \sim (N_p)$)

Research focus on reading off invariants (Clifford, geom) from

the #s a_{1j} (Green, Voisin, Aprodu-Farkas)

Strategy:

(i) Bound to ensure that sufficiently pos line bundles yields closed embeddings $\hookrightarrow \mathbb{P}$

(ii) find effective bound for (N_p) (Green)

(iii) For $(X, \mathcal{L})$ near but below the bound of (ii)

Show N_p fails $\Leftrightarrow \exists$ special structures on X

Higher dims:

(i) Try to make sure for given ample line bundle $\mathcal{L}$

$\omega_X \otimes \mathcal{L}^{\otimes m}$ is very ample for $m >> 0$

Conj (Fujita): X smooth proj var, $\mathcal{L}$ ample line bundle

$\omega_X \otimes \mathcal{L}^{\otimes m}$ is globally gen if $m \geq \dim X + 1$

very ample if $m \geq \dim X + 2$

(ii) Conj (Mukai)

$\omega_X \otimes \mathcal{L}^{\otimes m}$ satisfies (N_p) if $m \geq \dim X + p + 2$

Thm (Ein - Lazarsfeld)

$\mathcal{L}$ = v.a. line bundle

$(X, \mathcal{L}) \neq (\mathbb{P}, \mathcal{O}_{\mathbb{P}}(1))$ then

$\omega_X \otimes \mathcal{L}^{\otimes m}$ satisfies (N_p) for $m \geq \dim X + p$

Thm (~~Pat~~ Pareschi)

X = ab var, $\mathcal{L}$ = ample lbd on X

then $\mathcal{L}^{\otimes m}$ satisfies (N_p) for $m \geq p+3$

Problems:

(i) Bounds ~~doesn't~~ don't scale well

would be better for bounds in terms of int #'s.

(ii) ? Forbidden subvarieties

Both can be solved in the first case not

Thm (K, Lazarsfeld 15)

X = ~~ab line bundle~~ abelian surface

$\mathcal{L}$ = ample $p \in \mathbb{N}$

assume $(L^2) \geq \underbrace{5(p+2)^2}$ could be improved to 4 ~~by 4~~ by Ito

TFAE: (i) $\mathcal{L}$ doesn't sat (N_p)

(ii) $\exists$ ell curve $C \hookrightarrow X$. $(C^2)=0$

$(LC) \leq p+2$

Thm (Lazarsfeld, 18)

X abelian 3-fold

$\mathcal{L}$ ample

Idea of the proof:

- Green-Laz: reduce (N_p) to check vam of H^1 of vbds
- reduce further to vanishing of H^1(lbd $\otimes$ ideal sheaf) of $\underbrace{X \times \cdots \times X}_{p+2}$
- In the case of ab var.
- ....