Alterations

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Ref: - Abramovich, Ort Alterations

- de Jong, Smoothness, semistability and alterations

Thm (Hironaka, 64)

char $k=0$. X int f.t. scheme/k

$Z \xhookrightarrow{c.s.} X$, then there exists a finite seq of blowups (birational morphism)

$$X_n \to X_{n-1} \to X_{n-2} \to \cdots \to X_0 = X$$

s.t. X_n is regular

strict transform of Z is snc divisor.

Q: char p case?

A: Yes if we allow weaker conditions

Thm (de Jong, 1995)

X int f.t. /k. ~~Then~~ $Z \xhookrightarrow{c.s.} X$ closed subscheme then there exist an $X' \to X$ separable proper surjective and generically finite such that X' is regular & quasiproj and strict transform of Z is snc divisor.

Def.: $f: X' \to X$ is called an alteration if it's proper, surjective, generically finite.

Ex:

$$\begin{array}{l} X = \mathrm{Spec}\left(k[t][x,y]/(xy-t^2)\right) \quad (\mathrm{char}\, k \neq 2) \\ \downarrow \\ \mathbb{A}^1 = \mathrm{Spec}(k[t]) \end{array}$$

$x = (0,0,0)$ is the only singularity of X

How to resolve $x \in X$?

$Bl_x X$, then drop the exceptional divisor

$\rightsquigarrow$ smooth.

Rough idea:

$$\begin{array}{ccc} X' & \xrightarrow{\text{alteration}} & X \\ \downarrow{\scriptstyle \text{fibration of nodal curves.}} & & \\ P & & \end{array}$$

Proof (Sketch)

Step 1: (reduction)

Can "replace" (X, Z) by (X', Z')

s.t. X' is projective normal

& Z' is a divisor in X'

construction of $X' \to X$: (1) first blow-up at Z

$Bl_Z(X) \to X$

(2) Then Chow's lemma: $\exists$ a proper birational

$$X' \to Bl_Z(X) \quad \text{s.t. } X' \text{ is quasiprojective}$$

$$\begin{tikzcd} X' \arrow[r, hook, "\text{open-embedding}"] & \mathbb{P}^m \\ Z' \arrow[u, hook] \end{tikzcd}$$

so by taking closure of X' in $\mathbb{P}^m$

can assume X' is projective

(3) take the normalization of X'

<u>Step 2</u>: Find a projection

$$X \xhookrightarrow{c\text{-}i} \mathbb{P}^m$$

<u>Lemma</u>: (1) if $\dim X < m-1$, then $\exists$ open subset of pts in $\mathbb{P}^m$ s.t. projection from ~~this~~ each pt maps X birationally onto the image.

(2) if $\dim X = m-1$, then $\exists$ . . .

. . . .

X generically étale over $\mathbb{P}^{\dim X}$

$\Rightarrow X \xrightarrow{\pi} \mathbb{P}^d$ generically étale.

$(\dim X = d$

$\Rightarrow \begin{cases} B \subseteq \mathbb{P}^d \text{ s.t. } \pi \text{ is not regular on } B \\ \exists\, U \subseteq \mathbb{P}^d \text{ s.t. } \cancel{\pi(Z)} \text{ projection from pts in } U \text{ maps } \pi(Z) \text{ generically étale} \end{cases}$

Then choose pt $p \in U \setminus B$

take $Bl_{\pi^{-1}(p)} X = \{ (x, l) \subseteq X \times \mathbb{P}^{d-1} \mid \pi(x) \in l \}$

collection of lines going through $p \in \mathbb{P}^d$

$\rightsquigarrow Bl_{\pi^{-1}(p)}(X) \xrightarrow{f} \mathbb{P}^{d-1}$

$\Rightarrow$ "Check" this is a ~~rotutine~~ curve

<u>Step 3</u> Improvement of fiberation

$U \subseteq \mathbb{P}^{d-1}$ $f^{-1}(U)$ is smooth of dim 1 ~~over~~ over U

$\rightsquigarrow$ a family of nodal curve over U

$Sch \rightarrow Set$

$T \mapsto \{$ iso classes of family of nodal curves $/ T \}$

such functor is not representable.

$\rightsquigarrow$ rigidify the functor

Need family of <u>stable n-pointed</u> nodal curve of genus g

<u>Prop</u>. $\exists$ alteration of $P : P' \to P$

s.t.