The multiplier ideal is a universal test ideal (part 1)

Goal: (R, m) local, $\mathbb{Q}$-Gorenstein, normal
ess f.type / $k \supseteq \mathbb{Q}$

Then $\mathcal{J}(R) \bmod p = \tau(R \bmod p)$ $\forall p \gg 0$

<u>Need</u>: ① Reduction to positive char

② canonical covers

③ sort out canonical modules in CA & AG

① Start with: R eft$/k \supseteq \mathbb{Q}$

want: finite type $\mathbb{Z}$-alg $A \subseteq k$

and a subring $R_A \subseteq R$ eft$/A$

s.t. $R_A \otimes_A k \cong R$

Given such $A \subseteq R_A$

if $\mu \in \operatorname{MaxSpec}(A)$, then $\mathbb{Z} \to A/\mu$ must factor through $\mathbb{Z}/p\mathbb{Z}$ for some $p \in \mu \cap \mathbb{Z}$

In particular, $A/\mu \otimes_A R_A$ is a "char $p>0$" model for R
denote by R_p (or R_μ)

Reduction of projective morphisms

$X = \operatorname{spec}(R)$ and proj morphism

$$\begin{tikzcd} Y \arrow[r] \arrow[dr, "\pi"'] & \mathbb{P}^N_k \times X \arrow[d] \\ & X \end{tikzcd}$$

$$Y = \operatorname{Proj}\left(\frac{R[T_0, \dots, T_N]}{(G_1, \dots, G_l)}\right)$$

homog poly's

the coeff of G_i is a finite set

we can choose $R_A \subseteq R$ s.t. $G_i \in R_A$

take $$Y_A = \mathrm{Proj}\left(\frac{R_A[T_0, \dots, T_N]}{(G_1, \dots, G_l)}\right) \xrightarrow{\pi_A} \mathrm{Spec}(R_A)$$

If $Y \to \mathrm{Spec}(k)$ is smooth (e.g. $\pi: Y \to \mathrm{Spec}(R)$ is res of sings)

Then $Y_A \times_A \mathrm{Spec}(\mathrm{Frac}(A))$ is smooth

$\Rightarrow \exists\, U \neq \emptyset$, $U \subseteq \mathrm{Spec}(A)$ open

s.t. Y_p is smooth $\forall p \in U$.

If $\mu \in \mathrm{Max\,Spec}(A)$, $\mu \cap \mathbb{Z} = (p)$

Y_p is smooth $\forall\, p \gg 0$

Punch line: R eft $/k \supseteq \mathbb{Q}$

$$\exists\ R_A \leftarrow A \underset{\text{f.type}}{\overset{\text{flat}}{\longleftarrow}} \mathbb{Z} \ni p$$ and resolution of rings

for R, reduction them mod p.

③ canonical modules (CA vs. AG)

(R, m) normal local Noetherian (not necessarily complete) eft $/k$

$R = S/I$ where (S, n) is a regular local ring eft$/k$

$d = \dim R \quad n = \dim(S) \quad n - d = c$

$$\omega_S = \Lambda^n \Omega_{S/k} \cong S$$

<u>Goal</u>: $\mathcal{E}xt_S^c(R, \omega_S)$ has two properties:

3.1 If E_R = inj hull of R/m, then

$$H_m^d(R) \cong \mathrm{Hom}_R(\mathcal{E}xt_S^c(R, \omega_S), E_R)$$

<u>Thm</u> (local duality)

If M is a f.g. S-module, then

$$H_m^i(M) \cong \mathrm{Hom}_S(\mathcal{E}xt_S^{n-i}(M, \omega_S), E_S)$$

<u>Fact</u>: $R = S/I$, $\mathrm{Hom}_S(R, E_S) \cong E_R$

$$H_m^d(R) \cong \mathrm{Hom}_S(\mathcal{E}xt_S^{n-d}(R, \omega_S), E_S)$$

$$= \mathrm{Hom}_S(\mathcal{E}xt_S^{n-d}(R, \omega_S) \otimes_R R, E_S)$$

$$\cong \mathrm{Hom}_R(\mathcal{E}xt_S^{n-d}(R, \omega_S), \underset{\overset{\|}{E_R}}{\mathrm{Hom}_S(R, E_S)})$$

3.2 R is normal, $X = \mathrm{Spec}(R)$

$$X_{reg} \rightsquigarrow \Lambda^d \Omega_{X_{reg}/k} \cong \mathcal{O}_{X_{reg}}(K_{X_{reg}})$$

$$K_X = \overline{K_{X_{reg}}}$$

$$R(K_R) = \Gamma(X, \mathcal{O}_X(K_X))$$

is the ~~reflexi~~ reflexive module on X

giving $\mathcal{O}_{X_{reg}}(K_{X_{reg}})$

Lemma (Adjunction formula)

$R = S/I$ is regular

Then $\exists$ s.e.s $0 \to I/I^2 \to \Omega_S \otimes R \to \Omega_R \to 0$

apply Λ^{top} $\Rightarrow$ $\omega_S \otimes R \cong \omega_R \otimes \underbrace{(\Lambda^c I/I^2)}_{\text{free of rk 1 on } R}$

$$\Rightarrow \omega_R \cong (\omega_S \otimes R) \otimes (\Lambda^c I/I^2)^*$$

$$i^* \omega_S \otimes \Lambda^c N$$

where $i : \mathrm{spec}(R) \hookrightarrow \mathrm{spec}(S)$

If $R = S/fS$ and $\Delta = \mathrm{div}(f)$, then

$$K_R = (K_S + \Delta)|_R$$

Prop $\mathrm{Ext}^c_S(R, \omega_S)$ is a reflexive R-module.

Check has depth ≥ 2 using Koszul complex $\otimes G_\bullet \xrightarrow{\sim} R$

Prop If R is regular local, ~~then~~ $R = S/I$, then

$$\mathrm{Ext}^c_S(R, \omega_S) \cong i^* \omega_S \otimes \Lambda^c N \cong \omega_R$$

"Fundamental local isomorphism"

Residues & Duality Prop III 1.2

Pf: Since $R = S/I$ where $I = (x_1, \dots, x_c)$

where $x_1, \dots, x_c$ is part of a r.s. in S

Compute $\operatorname{Ext}^c_S(R, \omega_S)$ using $\operatorname{Kos}^S(\underline{x})$

$$0 \to \underset{e_1 \wedge \cdots \wedge e_c}{S} \xrightarrow{\begin{pmatrix} x_1 \\ \vdots \\ x_c \end{pmatrix}} S^c \to \cdots$$

apply $\operatorname{Hom}_S(-, \omega_S)$

$$\rightsquigarrow \cdots \to \bigoplus \operatorname{Hom}_S(S, \omega_S) \xrightarrow{(x_1, \dots, x_c)} \operatorname{Hom}_S(S, \omega_S) \to 0$$

Think of $\operatorname{Kos}^S_c(\underline{x})$ as $S(\varepsilon_1 \wedge \cdots \wedge \varepsilon_c)$

$$\operatorname{Hom}(S, \omega_S \otimes S(\varepsilon_1 \wedge \cdots \wedge \varepsilon_c)^*)$$

take cohomology

$$S/(\underline{x}) \otimes_S \omega_S \otimes_S S(\varepsilon_1 \wedge \cdots \wedge \varepsilon_c)^*$$

$$\cong (\omega_S \otimes R) \otimes_R \wedge^c \Big(\bigoplus_{i=1}^{c} R\varepsilon_i\Big)^*$$

$$\cong I/I^2$$

$$\operatorname{Ext}^c_S(R, \omega_S) \cong (\omega_S \otimes R) \otimes_R \wedge^c I/I^2$$

$$\underset{\text{adjunction}}{\cong} \omega_R$$