Kunz type characterization of regular rings

<u>Thm</u> (Kunz)

A ring of char $p > 0$ is regular $\iff$ Frobenius is flat

Suppose F is finite, A is local

then F being flat $=$ F_*A is a free A-module.

<u>Easy direction</u>: Suppose A is regular

(i.e. $A = k[[x_1, \cdots, x_n]]$, k is perfect)

then $\{x_1^{\lambda_1} \cdots x_n^{\lambda_n}\}_{0 \le \lambda_i \le p-1}$ is a basis.

Recall Auslander-Buchsbaum formula

(A, m) local, M a f.g. nonzero A-module of f.p dim

~~$pd(A) = pd(M) +$~~

$$\underset{\dim A}{\underset{\|}{\operatorname{depth}(A)}} = \underset{F_*A}{\underset{\uparrow}{pd(M)}} + \underset{\underset{\dim A}{\|}}{\underset{F_*A}{\underset{\uparrow}{\operatorname{depth}(M)}}}$$

$\Rightarrow pd(F_*A) = 0 \Rightarrow F_*A$ proj $\Rightarrow F_*A$ flat

Suppose (A, m) ess f. type / $\mathbb{C}$

is it true for a resolution of singularities

$$\pi : Y \to \operatorname{Spec} A$$

Consider $A \to R\pi_* \mathcal{O}_Y$

<u>Def</u> A has rational singularity

if $A \to R\pi_* \mathcal{O}_Y$ is a quasi-isom.

<u>Notice</u>: A^+ is a flat A-module when A is regular
(char $p > 0$)

<u>Thm</u>: A is regular $\Leftrightarrow$ A^+ is a flat A-module
in char $p > 0$

<u>Thm</u> (Ma, -) (A, m) ess f. type/$\mathbb{C}$. A is regular

$\Leftrightarrow$ $\forall \pi : Y \to \operatorname{Spec} A$ a regular alteration
$R\pi_* \mathcal{O}_Y$ has finite projective dimension

$\Leftrightarrow$ pd $R\pi_* \mathcal{O}_Y = 0$

$$A \subseteq S \to R\pi_* \mathcal{O}_Y$$

$$\pi_* I^0 \to \pi_* I^1 \to \pi_* I^2 \to \cdots$$

$$A^{\oplus n_1} \to A^{\oplus n_2} \to A^{\oplus n_3} \to \cdots$$

Def depth $(R\pi_* \mathcal{O}_Y)$ = smallest i

s.t. $\mathbb{H}^i_m(R\pi_* \mathcal{O}_Y) \neq 0$

[Roberts] $R\pi_* \mathcal{O}_Y$ is Cohen-Macaulay

$\mathbb{H}^{d-i}(R\pi_* \mathcal{O}_Y) = 0$ by GR-vanishing.

Lemma If for a resolution of sings

$$Y \to \operatorname{Spec} A$$

$R\pi_* \mathcal{O}_Y$ has finite proj dim

then A is Cohen-Macaulay.

$$\operatorname{depth} A = \underset{\underset{\underset{\dim A}{\shortparallel}}{\underset{d}{\shortparallel}}}{\operatorname{depth}(R\pi_* \mathcal{O}_Y)} + \underset{\underset{0}{\vee\!/}}{\operatorname{pd}(R\pi_* \mathcal{O}_Y)}$$

$$\cancel{\Rightarrow \operatorname{pd}(R\pi_* \mathcal{O}_Y) = 0}$$

$$\operatorname{depth} A \geq \dim A \Rightarrow A \quad \text{C-M}$$

Prop If $R\pi_* \mathcal{O}_Y$ has finite proj dim, then A has rational singularities

$(H^d_m(A) \to H^d_m(R\pi_* \mathcal{O}_Y)$ is an isom$)$

Pf: WLOG we may assume A has rational sings on punctured spec.

$$\pi_* \omega_Y \hookrightarrow \omega_A$$

WTS this is an isom.

~~Thm (Sharp)~~

$$\pi_* \omega_Y = R\pi_* \omega_Y = R\mathcal{H}om(R\pi_* \mathcal{O}_Y, \omega_A)$$

has finite injective dim

<u>Thm</u> (Sharp) $\mathcal{H}om(\omega_A, \pi_* \omega_Y)$ has finite proj dim.

$$\mathcal{H}om(\omega_A, \pi_* \omega_Y) \subseteq \mathcal{H}om(\omega_A, \omega_A) \cong A$$

m-primary

Aim to show $\mathcal{H}om(\omega_A, \pi_*\omega_Y)$ is int closed

$$\mathcal{H}om(\omega_A, \pi_* \omega_Y) = \pi_* \mathcal{H}om(\pi^* \omega_A, \omega_Y)$$

locally free of rank 1

push-forward of a rank 1

line bundle is int closed

So A is regular $\Rightarrow$ A is rational contradiction!

So A has rational singularity.

<u>Prop</u> Suppose (A, m) is a normal local domain

ess f.type / $\mathbb{C}$. $0 \neq f \in A$, $\sqrt{(f)} = (f)$

consider $S = A[f^{\frac{1}{N+1}}]$

$\mathcal{J}(\omega_S)$ has a summand $\cong \mathcal{J}(\omega_A, f^{\frac{N}{N+1}})$

$\quad \Vert$

$p_* \omega_Z$

where $p: Z \to \operatorname{Spec} S$

Pf of main thm:

Choose $N > 0$ s.t.

$\mathcal{J}(\omega_A, m^N) \neq \omega_A$

Choose $f \in m^{N+1}$ generally

$\mathcal{J}(\omega_A, f^{N/N+1})$ summand of $\mathcal{J}(\omega_S)$