# Singularities in mixed char

Def: $A$ stalk at a point $p \in X$

$(A, m)$ a local ring essentially of finite type $/\mathbb{C}$.

$(A, m)$ has rational singularities if for some/any resolution $\pi : Y \to \operatorname{spec} A$

① $\pi_* \mathcal{O}_Y = A$

② $R^i \pi_* \mathcal{O}_Y = 0 \quad \forall\, i > 0$

If $X$ has rational singularity at all points

then $H^i(X, \mathcal{L}) = H^i(Y, \pi^* \mathcal{L})$

- $\mathbb{C}[x, y, z] / (x^n + y^n + z^n)$ has rational singularity if $n \leq 2$
- $\mathbb{C}[x_1, \dots, x_m] / (x_1^n + \dots + x_m^n) \Rightarrow$ rational singularity $\Leftrightarrow n < m$

<u>Alternate characterisation</u>

$A$ has rational singularitis

$\Leftrightarrow H^i_m(A) \xrightarrow{\sim} \mathbb{H}^i_m(R\pi_* \mathcal{O}_Y)$

$\forall\, i < d = \dim A$, $\mathbb{H}^i_m(R\pi_* \mathcal{O}_Y) = 0$ $\qquad Y \xrightarrow{\pi} \operatorname{spec} A$

$\updownarrow$ always true. $\qquad \updownarrow$

Grauert-Riemenschneider $\qquad R^{d-i} \pi_* \omega_Y = 0$

so if $A$ has rational singularity

then $H^i_m(A) = 0$ i.e. $A$ is CM.

<u>Thm</u> (Kempf)

A has rational singularities iff

① A is C-M

② $H^d_m(A) \xrightarrow{\sim} \mathbb{H}^d_m(R\pi_*\mathcal{O}_Y)$

<u>Thm</u> (Elkik) If $(A, m)$, $f \in A$ / $\mathbb{C}$ not a zerodivisor. If $A/f$ has rational singularities then so does A.

<u>Pf</u>: Since $A/(f)$ is C-M, so is A

apply local cohomology to $0 \to A \xrightarrow{f} A \to A/fA \to 0$

$$\begin{tikzcd}
0 \arrow[d, equal] & \rightsquigarrow\ 0 \arrow[r] & \mathbb{H}^{d-1}_m(A/f) \arrow[r] \arrow[d, hook] \arrow[dd, bend right, "\text{isom}"'] & H^d_m(A) \arrow[r, ".f"] \arrow[d] & H^d_m(A) \arrow[d] \\
H^{d-1}_m(R\pi_*\mathcal{O}_Y) \arrow[rr] & & H^{d-1}_m(R\pi_*\mathcal{O}_{\bar H}) \arrow[r, hook] \arrow[d, hook] & H^d_m(R\pi_*\mathcal{O}_Y) \arrow[r, ".f"] & H^d_m(R\pi_*\mathcal{O}_Y) \\
& & H^{d-1}_m(R\pi_*\mathcal{O}_{\tilde H}) & &
\end{tikzcd}$$

~~where $\bar H = \pi^* \mathrm{div}(f)$~~

$$\begin{tikzcd}
\bar H = \pi^* \mathrm{div}(f) \arrow[dr] & \\
\tilde H \arrow[u] \arrow[r] & \mathrm{Spec}\, A/f
\end{tikzcd}$$

resolution

Defn If $(A, m)$ is $\mathbb{Q}$-Gorenstein
then $(A, m)$ is log terminal
$\Leftrightarrow$ a canonical cover has rational singularities

we also have $(A, m)$ $\mathbb{Q}$-Gor, $A/f$ l.t. $\Rightarrow A$ l.t.

Observation/$\mathbb{C}$

$R\pi_* \mathcal{O}_Y$ is "Cohen-Macaulay" (F-finite)

char $p>0$: Suppose $(A, m)$ is a local ring of char $p>0$

Defn: $A$ is F-rational domain
① $A$ is CM
② $H^d_m(A) \hookrightarrow H^d_m(A^+)$ ($A^+$ is the ~~abs~~ absolute int closure)

Thm (Smith, ~~Hara~~, Mehta-Srinivas)
If $(A, m)$ is in char $0$, then $A$ has rational singularities $\Leftrightarrow$ $(A \bmod p, m_p)$ has F-rational singularity $\forall p \gg 0$

Mixed characteristic

- char $A = 0$
- char $A/_m = p > 0$
- No resolution of rings (yet?)
- No GR-vanishing
- $A^+$ is not CM

Thm (André) $(A, m)$ complete local ring,

$\exists$ a ~~bcu~~ big CM perfectoid $A^+$- algebra, $B$

Def. $(A, m)$ fix a perfectoid big CM alg $B$

$A$ is $PBCM_B$ - rational if

① $A$ is CM

② $H^d_m(A) \hookrightarrow H^d_m(B)$

Def. $(A, m)$ is PBCM-rational if it is $PBCM_B$-rational for all $B$.

Thm If $S$ is a set of perfectoid BCM $A^+$ algebras $\{B_i\}$ then $\exists$ a perfectoid BCM $A^+$-alg $B$ s.t. $B_i \to B$ as $A^+$-alg.

Thm (Ma, –) If $(A, m)$ is PBCM-rational, then

$\forall \pi : Y \to X$ proper birational $(X = \operatorname{spec} A)$

then $H^d_m(A) \to H^d_m(R\pi_* \mathcal{O}_Y)$ is an injection

In char $p > 0$, $R$ is F-rational iff

① $R$ is CM

② $\forall B$ BCM, $H^d_m(R) \hookrightarrow H^d_m(B)$

Thm (Ma, –) If $(A, m)$ is a complete local

$f \in A$ is a nonzero divisor

$A/f$ is PBCM-rational

then $A/f$ is PBCM-rational

Thm $(A, m)$ in char $0$ is such that $(A \bmod p, m \bmod p)$

is F-rational for a single $p$. then

$A$ has rational singularities

Need: $A_{\mathbb{Z}}$ with $p$ NZD on $A_{\mathbb{Z}}$

$m_{\mathbb{Z}} + (p) \neq A_{\mathbb{Z}}$

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If $A$ has rational singularities in char $0$,

is it true $\forall$ BCM $A$-alg $H^d_m(A) \hookrightarrow H^d_m(B)$