~~Multipli~~ Multiplier ideals

X smooth complex variety

$\mathfrak{a} \subset \mathcal{O}_X$

D an effective $\mathbb{Q}$-divisor $\quad [D]$ = round down / floor of D

$t \geq 0$ real number

Def - The multiplier ideal sheaf associated to D

$\mathcal{J}(X, D)$ is defined as follows:

~~$\mathcal{J}(D)$~~ Let $f: Y \to X$ be a log resolution of D

($\tilde{D}$ to be smooth, $\tilde{D} + Exc(f)$ snc)

$$\mathcal{J}(D) = f_* \mathcal{O}_Y(\underbrace{K_{Y/X} - [f^* D]}_{\text{integral divisor}})$$

Analogously: Let $f: Y \to X$ be a log resolution of $\mathfrak{a}$

$\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$ $\quad A$ an effective divisor on Y

$$\mathcal{J}(\mathfrak{a}^t) = f_* \mathcal{O}_Y(K_{Y/X} - [t \cdot A])$$

Remarks: (1) check this is independent of $f: Y \to X$ as a log resolution

(2) $$\underset{\mathcal{O}_X}{\underset{\cong}{f_* \mathcal{O}_Y(K_{Y/X})}} = f_*(\mathcal{O}_Y(K_Y - f^* K_X)) = \mathcal{O}_X(-K_X) \otimes \underbrace{f_* \mathcal{O}_Y(K_Y)}_{\mathcal{O}_X(K_X)}$$

Example: $X = \mathrm{Spec}(R)$ D a divisor, compute $\mathcal{J}(tD)$

$Y \to X$ as log resolution

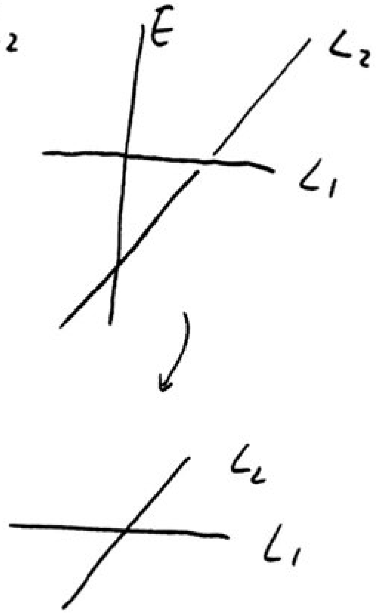
- $K_{Y/X} = \sum k_i E_i$

  $f^*D = \sum a_i E_i \quad a_i \geq 0$

- $K_{Y/X} - [t f^* D] = \sum (k_i - [t a_i]) E_i$

  $$\mathcal{J}(tD) = \{ f \in \mathrm{Frac}(R) : \underbrace{\mathrm{div}_Y(f) + K_{Y/X} - [t f^* D] \geq 0}_{\mathrm{ord}_{E_i}(f) + k_i - [t a_i] \geq 0} \}$$

$X = \mathbb{A}^2 \quad D = V(xy) \quad L_1 + L_2$



$Y = Bl_{(0,0)} \mathbb{A}^2$

$K_{Y/X} = E$

$f^*D = \tilde{L}_1 + \tilde{L}_2 + 2E$

$$K_{Y/X} - [f^*(tD)] = (1 - 2[t])E + [t](\tilde{L}_1 + \tilde{L}_2)$$

$f \in k[x,y]$, write $f = x^a y^b \cdot f_0$

$\mathrm{ord}_E(f) = a + b + \cancel{\deg(f_0)}\ \mathrm{mult}_0(f_0)$

$\mathrm{ord}_{\tilde{L}_1}(f) = a$

$\mathrm{ord}_{\tilde{L}_2}(f) = b$

$$f \in \mathcal{J}(tD) \text{ iff } \begin{cases} 1-2[t]+a+b+\mathrm{mult}_0(f_0) \geq 0 \\ a-[t] \geq 0 \\ b-[t] \geq 0 \end{cases}$$

$$\Leftrightarrow \quad a, b \geq [t]$$

$$\mathcal{J}(tD) = (xy)^{[t]}$$

<u>Remark</u>: If A is an integral divisor on X,

then $\mathcal{J}(A) = \mathcal{O}_X(-A)$

$$f^*(K_{Y/X} - f_*A) = \mathcal{O}_X(-A) \otimes \mathcal{O}_X$$

$(\mathbb{A}^2, D = V(x^2-y^3))$

$$\mathcal{J}(\mathbb{A}^2, tD) = \begin{cases} (1) & 0 < t < \frac{5}{6} \\ (x,y) & \frac{5}{6} \leq t < 1 \\ (x^2-y^3) & 1 \leq t < 11/6 \\ (x,y)\cdot(x^2-y^3) & \frac{11}{6} < t \leq 2 \end{cases}$$

$$\mathrm{lct}(X, D) = \min_t \mathcal{J}(X, tD) \neq \mathcal{O}_X$$

(equiv, largest t s.t. (X, tD) log canonical)

$$\mathrm{lct}_x(X, D) = \min_t (\mathcal{J}(tD)_x \neq \mathcal{O}_{X,x})$$

Thm Let $\mathfrak{a} \subset \mathcal{O}_X$, X smooth of dim n

$$\overline{\mathfrak{a}^m} \subset \mathfrak{a}^{m-n+1} \quad \text{if} \quad m \geq n$$

[Briançon - Skoda]

Thm [Skoda's]

If $m \geq n$, $\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a} \cdot \mathcal{J}(\mathfrak{a}^{m-1})$

so $\mathcal{J}(\mathfrak{a}^m) = \mathfrak{a}^{m-n+1} \mathcal{J}(\mathfrak{a}^{n-1})$

Pf of Briançon - Skoda using Skoda:

$\mathcal{J}(\mathfrak{a}^t)$ is integrally closed for any t

$$\mathfrak{a} \subset \mathcal{J}(\mathfrak{a}) \Rightarrow \overline{\mathfrak{a}} \subset \mathcal{J}(\mathfrak{a})$$

$\#$

Thm Let $f: Y \to X$ be a log resolution of $\mathfrak{a}$: $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-A)$

$$R^i f_* \mathcal{O}_Y (K_{Y/X} - [tA]) = 0 \quad \text{for } i > 0$$