A stable version of Harbourne's Conjecture

R regular ring

I radical ideal

h big height of I

$= \max\{ \operatorname{ht} Q : Q \in \operatorname{Ass}(R/I) \}$

n^{th} symbolic power of I

$$I^{(n)} = \bigcap_{Q \in \operatorname{Min}(R/I)} (I^n R_Q \cap R)$$

If I is generated by a regular sequence, then

$$I^{(n)} = I^n \quad \forall n \geqslant 0$$

<u>Ex</u>: $I = (xy, yz, xz) \subseteq K[x,y,z]$

$= (x,y) \cap (x,z) \cap (y,z)$

$I^{(2)} = (x,y)^2 \cap (x,z)^2 \cap (y,z)^2$

$xyz \in I^{(2)}$ but $xyz \notin I^2$ ← all deg $\geqslant 4$

Containment Problem: when is $I^{(a)} \subseteq I^b$?

<u>Thm</u> (Ein - Lazarsfeld - Smith, 2001,
Hochster - Huneke, 2002,
Ma - Schwede, 2017)

$$I^{(hn)} \subseteq I^n \Rightarrow I^{(dn)} \subseteq I^n \quad \forall n \geqslant 1$$

Ex $P = \ker(k[x,y,z] \to k[t^3,t^4,t^5])$

$$P^{(2)} \neq P^2 \qquad P^{(3)} \subseteq P^2 \qquad (\text{thm} \Rightarrow) \; P^{(4)} \subseteq P^2$$

Question: If P prime of height 2 in RLR, is $P^{(3)} \subseteq P^2$?

Conjecture: (Harbourne, 2008) (✯)

$$I^{(hn-h+1)} \subseteq I^n \quad \forall n \geq 1.$$

Remark (Hochster - Huneke)

In char $p>0$, $q = p^e$

$$I^{(hq-h+1)} \subseteq I^{[q]}$$

Ex (counter example to ✯) [2013]

- $I = (x(y^3-z^3), y(z^3-x^3), z(x^3-y^3)) \subseteq \mathbb{C}[x,y,z]$

  $h=2$

  $I^{(3)} \nsubseteq I^2$

- $I = (x(y^c-z^c), y(z^c-x^c), z(x^c-y^c)) \subseteq k[x,y,z]$

  $\text{char}(k) \neq 2$

  $h=2$

  and $I^{(3)} \nsubseteq I^2$ $\quad (c \geq 3)$

Harbourne's conjecture does hold

- General points in $\mathbb{P}^2$ (Ha Hu)

  and in $\mathbb{P}^3$ (Dumnicki)

- In char $p>0$,

  if R/I is F-pure (G-Huneke, 2017)

<u>Example</u>: $I = \bigcap_{i \neq j} (x_i, x_j) \subseteq k[x_1, \cdots, x_d]$

$$= (x_1 \cdots \hat{x}_i \cdots x_d,\ 1 \leq i \leq d)$$

$h=2$: $I^{(2n-1)} \subseteq I^n$ ✓

Q: $I^{(2n-2)} \subseteq I^n$? No for $n<d$

$$I^{(2n-2)} = \bigcap_{i \neq j} (x_i, x_j)^{2n-2}$$

$$\cup\!\!\!\!\!\shortmid$$

$$(x_1 \cdots x_d)^{n-1}$$

deg of elements here

$\hookrightarrow$ $\deg = dn - d < dn - n = n(d-1)$

If $n<d$

<u>Harbourne's Conjecture</u> (stable version)

$$I^{(hn-h+1)} \subseteq I^n \text{ for } n \gg 0$$

<u>Thm</u> R containing a field, If $I^{(hk-h)} \subset I^k$ for some k

then $I^{(hn-h)} \subseteq I^n$ for $n \gg 0$

"Resurgence" (Bocci- Harbourne)

$$\rho(I) = \sup\left\{ \frac{a}{b} : I^{(a)} \nsubseteq I^{b} \right\}$$

then $1 \leq \rho(I) \leq h$

If $\rho(I) < h$:

$$\frac{hn-h+1}{n} > \rho(I) \iff I^{(hn-h+1)} \subseteq I^{n}$$

$$\Updownarrow$$

$$n > \frac{h-1}{h-\rho(I)}$$

$$\frac{hn-c}{n} > \rho(I) \iff I^{(hn-c)} \subseteq I^{n}$$

$$\Updownarrow$$

$$n > \frac{c}{h-\rho(I)}$$

Question: ① $I^{(hn-c)} \subseteq I^{n}$ for $n \gg 0$

② Is there ideal I with $\rho(I) = h$?

Q: $I^{(n+h)} \subseteq I \cdot I^{(n)}$ for all $n \geq 1$

Not true for all I: there's I s.t. $h=2$, $n=1$, $I^{(3)} \nsubseteq I^{2}$

Not ~~true~~ always true for all $n \gg 0$ (Seceleanu)

Thm: If R/I is F-pure then $I^{(n+h)} \subseteq I \cdot I^{(n)}$