

Log discrepancies and Frobenius splittings

① Semivaluations

Defn ① A valuation ~~on~~ ^{on} a field L is a group

$$\text{hom } v: L^\times \rightarrow (\mathbb{R}, +)$$

$$\text{s.t. } v(\cancel{f} + g) \geq \min \{v(f), v(g)\}$$

$$v(0) = +\infty$$

② R is a ring, A semivaluation on R

is a valuation on $K(P) = (R/P)_P$

for some $P \in \text{Spec}(R)$

③ A semivaluation φ on R is centered on R

$$\text{if } \varphi(f) \geq 0 \quad \forall f \in R$$

~~The~~ The center of such φ is

$$C_R(\varphi) = \underbrace{\varphi^{-1}(0, +\infty]}_{\cap R_P} \cap R$$

② Log discrepancies in char $p > 0$

R is reduced ring of char $p > 0$

Defn: Let \mathcal{V} be a semivaluation centered on R

[C.] $\text{Ker}(\mathcal{V}) = P$

$$C_R(\mathcal{V}) = Q$$

The log discrepancy of \mathcal{V} on R is

$$A_R(\mathcal{V}) = \sup_{\substack{\mathbb{R} \\ [-\infty, +\infty]}} \left\{ \frac{1}{q-1} \mathcal{V}(f) \mid \begin{array}{l} f \in R_P, \mathcal{V}(\psi(f^{1/q})) = 0 \\ \text{for some } q = p^e \\ \psi: R_Q^{1/q} \rightarrow R_Q \end{array} \right\}$$

Ex: $R = \mathbb{F}_p[[t]]$ $R^{1/p} \cong \bigoplus_{j=0}^{p-1} R t^{j/p}$

$$\Phi_R(\sum r_j t^{j/p}) = r_{p-1}$$

$\psi: R^{1/p} \rightarrow R$ any R -linear map

$$\psi(f^{1/p}) = \Phi_R(s \psi^{1/p} f^{1/p}) \text{ where there is some unique } s \in R$$

if $\text{ord}_t(\psi(f^{1/p})) = 0$

$$\Rightarrow \text{ord}_t(s) + \text{ord}_t(f) \leq p-1$$

$$\Rightarrow \frac{\text{ord}_t(f)}{p-1} \leq 1 - \frac{\text{ord}_t(s)}{p-1}$$

log discrepancy of Δ_ψ on ord_t

$$\therefore A_p(\text{ord}_t) = 1 \quad \text{use } \mathbb{F}_p$$

Ex: $R = \text{reduced}$ char $p > 0$ domain

and ~~\exists~~ $\exists \psi: R^{1/p} \rightarrow R$ nonzero

and \mathcal{V} is a semi-val with $\ker \mathcal{V} = \mathfrak{p} \neq (0)$

suppose $\psi(R^{1/p}) \not\subseteq \mathfrak{p}$

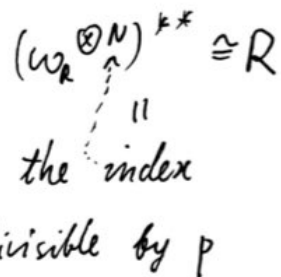
$$\Rightarrow \psi_p(R^{1/p}) = R_p$$

$$\Rightarrow \exists f \in \mathfrak{p} \text{ s.t. } \psi_p(f^{1/p}) = 1$$

$$\mathcal{V}(\psi_p(f^{1/p})) = 0 \Rightarrow A_R(\mathcal{V}) \geq \frac{1}{p-1} \mathcal{V}(f) = +\infty$$

THM [HW]

(R, m) F -finite, normal, \mathbb{Q} -Gorenstein with the index not divisible by p



Then

① R is F -split $\Rightarrow A_R(v) \geq 0 \quad \forall$ valuation v centered on R

② R is strongly F -regular $\Rightarrow A_R(v) > 0$

\forall nontrivial valuation v centered on R

($\text{triv}_R : \text{Frac}(R) \rightarrow \{0\}$)

THM: [C.]

$R = F$ -finite, Noetherian

① R is F -split iff $A_R(\mathcal{S}) \geq 0 \quad \forall$ semi-valuation centered on R

② R is strongly F -regular iff $A_R(\mathcal{S}) > 0 \quad \forall$ nontrivial semi-valuation centered on R

Pf: localize, look at $A_R(\text{triv}_p)$

for $\mathfrak{p} =$ splitting prime

where $\text{triv}_p: K(\mathfrak{p})^* \rightarrow \{0\} \quad \#$

Q: Suppose R is F -pure not strongly F -regular

is there a semi-valuation $\mathcal{S} \neq \text{triv}_p$ with $A_R(\mathcal{S}) = 0$?