Log discrepancies and Frobenius splittings

(I) Semivaluations

__Defn__ ① A valuation on ~~of~~ a field L is a group hom $v : L^* \to (\mathbb{R}, +)$

s.t. $v(f+g) \geq \min\{v(f), v(g)\}$

$v(0) = +\infty$

② R is a ring, A semivaluation on R is a valution on $K(P) = (R/P)_P$ for some $P \in \mathrm{spec}(R)$

③ A semivaluation φ on R is centered on R if $\varphi(f) \geq 0 \quad \forall f \in R$

~~Then~~ The center of such φ is

$$C_R(\varphi) = \underbrace{\varphi^{-1}(0, +\infty]}_{\substack{\cap\mid \\ R_P}} \cap R$$

(II) Log discrepancies in char $p>0$

R is reduced ring of char $p>0$

<u>Defn</u>: Let φ be a semivaluation centered on R

[C.] $$\mathrm{Ker}(\varphi) = P$$
$$C_R(\varphi) = Q$$

The log discrepancy of φ on R is

$$\underset{[-\infty,+\infty]}{\underset{\cap}{A_R(\varphi)}} = \sup\left\{ \frac{1}{q-1}\varphi(f) \;\middle|\; \begin{array}{l} f\in R_P,\ \varphi(\psi(f^{1/q}))=0 \\ \text{for some } q=p^e \\ \psi: R_Q^{1/q} \to R_Q \end{array} \right\}$$

<u>Ex</u>: $R=\mathbb{F}_p[[t]]$ $\qquad R^{1/p} \underset{R}{\cong} \bigoplus_{j=0}^{p-1} R\, t^{j/p}$

$$\Phi_R\left(\sum r_j t^{j/p}\right) = r_{p-1}$$

$\psi: R^{1/p} \to R$ any R-linear map

$\psi(f^{1/p}) = \Phi_R(s_\psi^{1/p} f^{1/p})$ where there is some unique $s_\psi \in R$

if $\mathrm{ord}_t(\psi(f^{1/p}))=0$

$\Rightarrow \mathrm{ord}_t(s_\psi) + \mathrm{ord}_t(f) \le p-1$

$$\Rightarrow \frac{\mathrm{ord}_t(f)}{p-1} \le \underbrace{1-\frac{\mathrm{ord}_t(s_\psi)}{p-1}}_{\text{log discrepance of } \Delta_\psi \text{ on } \mathrm{ord}_t}$$

$\therefore A_R(\mathrm{ord}_t) = 1$ use Φ_R

<u>Ex</u>: $R =$ ~~reduced~~ char $p > 0$ domain

and ~~Φ~~: $\exists \psi : R^{1/q} \to R$ nonzero

and ς is a semi-val with $\ker \varsigma = P \neq (0)$

suppose $\psi((PR^{1/q})) \notin P$

$\Rightarrow \psi_P(PR_P^{1/q}) = R_P$

$\Rightarrow \exists f \in P$ s.t. $\psi_P(f^{1/p}) = 1$

$\varsigma(\psi_P(f^{1/q})) = 0 \Rightarrow A_R(\varsigma) \geq \frac{1}{q-1}\varsigma(f) = +\infty$

<u>THM</u> [HW]

(R, m) F-finite, normal, $\mathbb{Q}$-Gorenstein with the index not divisible by p

$(\omega_R^{\otimes N})^{**} \cong R$

Then

① R is F-split $\Rightarrow A_R(v) \geq 0$ $\forall$ valuation v centered on R

② R is strongly F-regular $\Rightarrow A_R(v) > 0$

$\forall$ nontrivial valuation v centered on R

$(\mathrm{triv}_R : \mathrm{Frac}(R)^* \to \{0\})$

THM: [C.]

R = F-finite, Noetherian

① R is F-split iff $A_R(\varphi) \geq 0$ $\forall$ semi-valuation centered on R

② R is strongly F-regular iff $A_R(\varphi) > 0$ $\forall$ nontrivial semi-valuation centered on R

Pf: localize, look at $A_R(\mathrm{trivial}_p)$
for p = splitting prime
where $\mathrm{triv}_p : K(P)^* \to \{0\}$ #

Q: Suppose R is F-pure not strongly F-regular
is there a semivaluation $\varphi \neq \mathrm{triv}_p$ with $A_R(\varphi) = 0$?