On ideals defining products of closed subschemes of affine spaces (joint w/ Irana Swanson)

Robert Walker

Consider a proper ideal I in R (Noetherian)

Def. The stable assassinator of I is

$$\mathcal{A}_R(I) = \bigcup_{c=1}^{\infty} \mathrm{Ass}_R(R/I^c)$$

Thm (Brodmann, 1979) $\# \mathcal{A}_R(I) < \infty$

Indeed, $\mathrm{Ass}_R(R/I^c)$ stabilizes for all $c \gg 0$

Def. I has the persistence property if $\forall c \geq 1$

$$\mathrm{Ass}_R(R/I^c) \subseteq \mathrm{Ass}_R(R/I^{c+1})$$

Thm (Katz & Ratliff, 1988)

Any nonzero normal ideal in a Noetherian domain has the persistence property.

$(I^c = \overline{I^c}, \forall c)$

Def: $P \subset R$ prime, $P^{(c)} := P^c R_P \cap R$

- $P^{(0)} = P^0 = R$
- $P^{(1)} = P$
- $P^c \subseteq P^{(c)}$

Ex: Let $R = \mathbb{C}[x,y,z]/(xy, xz, yz)$ & $P = (x,y)R$

then $P^c \subseteq P^{(c)} = P \quad \forall c \geq 1.$

basic setup

work with polynomial rings over a field k

and fix ideals $I \subseteq A = k[x_1, \cdots, x_m]$ &

$$J \subseteq B = k[y_1, \cdots, y_n]$$

Let $C = A \otimes_k B$, IC & JC

$$K = IC + JC \leftrightarrow V(I) \times_{\mathrm{Spec}\, k} V(J)$$

<u>Questions</u>: (1) Express $\mathcal{A}_C(K)$ in terms of

$$\mathcal{A}_C(IC) = \mathcal{A}_A(I) \;\&\; \mathcal{A}_C(JC) = \mathcal{A}_B(J)$$

(2) Conditions when persistance prop holds for K

(3) Express $K^{(c)}$ in terms of $I^{(a)}, J^{(b)}$ with $a, b \leq c$

(4) Express primary decomposition of K^c in terms of corresponding data for I^a & J^b

Our answers:

(1) $$\mathcal{A}(K) = \bigcup_{\substack{P \in \mathcal{A}(I) \\ Q \in \mathcal{A}(J)}} \{ P \in \mathrm{Min}((C/_{PC + QC})) \}$$

(2) If I or J is nonzero + normal, then K has persistence property

(3) $\forall\, c \geq 0$, $$K^{(c)} = \sum_{a+b=c} I^{(a)} J^{(b)}$$

<u>Ex</u>: If I & J are principal, then $I^{(a)} = I^a$, $\cancel{I^{(b)}}\ J^{(b)} = J^b$ $\forall\, a, b$

Def A list of ideals $\{L_c\}_{c\geq 0}$ in C is a filtration if $L_0 = C$ & $L_c \supseteq L_{c+1}$ $\forall c \geq 0$

Def Suppose $\mathcal{A}(I, C) = \{P_1, \cdots, P_r\}$

$$\mathcal{A}(J, C) = \{Q_1, \cdots, Q_s\}$$

For each $c \geq 1$, fix primary decomposition

$$(IC)^c = P_{c,1} \cap \cdots \cap P_{c,r}$$

$$\& \ (JC)^c = q_{c,1} \cap \cdots \cap q_{c,s}$$

s.t. $\forall 1 \leq i \leq r$, $P_{c,i}$ is either P_i-primary or C

& $\forall 1 \leq j \leq s$, $q_{c,j}$ is either Q_j-primary or C

Lem $\{P'_{c,i} = \bigcap_{a=0}^{c} P_{a,i}\}_{c=0}^{\infty}$

(4) Over an algebraically closed field k:

$$\forall c \geq 0, \quad K^c = \bigcap_{i=1}^{r} \bigcap_{j=1}^{s} \underbrace{\left(\sum_{a=0}^{c} P'_{a,i} \cdot q'_{c-a,j} \right)}_{L_{c,i,j}}$$