Valuation Rings

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- Applications
- Examples of non-Noetherian Rings

Ex: Consider $\mathbb{C}[[t]] \subset \mathbb{C}((t)) =: K$

Recall $f \in K$ can be written as $f = \sum_{i=N}^{\infty} a_i t^i$ $(N \in \mathbb{N})$

we can define $v_t(f) = N$

$v_t(t) = 1, \quad v_t(t^{-2} + 5t + 3) = -2$

For f, g $\bullet$ $v(fg) = v(f) + v(g)$

$\bullet$ $v(f+g) \geq \min\{v(f), v(g)\}$

set $v(0) = \infty$

$\bullet$ $v(f) = \infty$ iff $f = 0$

Then v is called a valuation

So $\mathbb{C}[[t]]$ is a valuation ring of $\mathbb{C}((t))$ under v_t with following property

- unique maximal ideal $m = \{f \mid v_t(f) > 0\}$
- $v(f) \leq v(g)$ iff $f \mid g$ in $\mathbb{C}[[t]]$
- $\forall f \in \mathbb{C}((t))$, either f or $\frac{1}{f}$ is in $\mathbb{C}[[t]]$

<u>Ex</u> (p-adic int) $v_p : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$

$$p^e \cdot \frac{a}{b} \mapsto e$$

$$0 \longmapsto \infty$$

<u>Def 1</u>. K a field, $R \subset K$ valuation ring of K

if $\forall\, x \neq 0 \in K$, either x or $x^{-1} \in R$.

If R is a domain, then R is valuation ring

if R is inside $\mathrm{Frac}(R)$

<u>Def 2</u>: A totally ordered abelian group Γ

is an abelian group such that

$\forall\, a \leq b \in \Gamma$, $\quad a+c \leq b+c \in \Gamma$

Let K be a field and Γ a totally .... group

A Γ-valuation of K is a ~~fraction~~

function: $v\ K \to \Gamma \cup \{\infty\}$

- $v(fg) = v(f) + v(g)$
- $v(f+g) \geq \min(v(f), v(g))$
- $v(f) = \infty$ iff $f = 0$

we require $\cancel{v}$ v to be surjective, Hence Γ is

called the valuation group of v

valuation ring $= \{ f \in K \mid v(f) \geq 0 \}$

<u>Rmk</u>: Let $R \subset K$ be a valuation ring

we have an inclusion $R^* \subset K^*$

Set $\Lambda = K^*/R^*$, we have

a surjection $K^* \to \Lambda$ extends to $K^* \to \Lambda \cup \{\infty\}$

<u>Ex</u>, $S = \mathbb{C}((t))$

$R = \mathbb{C}[[t]]$

$R^* = \{a_0 + a_1 t + \cdots \mid a_0 \neq 0\}$

$\Rightarrow S^*/R^* \cong \{t^l\}$ multiplicative valuation.

and the order is given by

$t^l > t^k$ if $t^l/t^k \in R$

<u>Ex</u>: $\cdot\, v(x) = 0$, $v(y) = 1$ on $K[x,y]$

$\Rightarrow k[x,y]_{(y)}$ valuation ring in $\mathrm{Frac}(K[x,y])$

$\cdot\, v(x) = v(y) = 1$

$\Rightarrow R_v = k[x, \frac{y}{x}]_{(x)}$ the exceptional divisor of the ~~bot~~ blow-up of $\mathbb{A}^2$ at $(0,0)$

Let $R \subset K$ be a valuation ring : $K \xrightarrow{v} \Gamma \cup \{\infty\}$

- R is local with maximal ideal $\{f \in R : v(f) > 0\}$

- ideals are totally ordered: given $I_1, I_2 \subset R$,
  either $I_1 \subset I_2$ or $I_2 \subset I_1$

- finitely generated ideals are principal

Hence in a Noetherian valuation ring, every idea is principal, i.e. it's a PID

If R is a Noetherian valuation ring, π is a uniformizer
i.e. $(\pi)R = m$

<u>Check</u>: $v: K \to \mathbb{Z} \cup \{\infty\}$

such rings are called DVRs.

<u>Ex</u>: Let $K = \bigcup_n \mathbb{C}((t^{1/n}))$

(<u>Check</u>: This is an algebra closure of $\mathbb{C}((t))$)

Now we can define a valuation $v: K \to \mathbb{Q} \cup \{\infty\}$ as usual

$R = \bigcup_n \mathbb{C}[[t^{1/n}]]$, $m = (t^{1/n} : n \in \mathbb{Z})$ <u>check</u> $m = m^2$

this is true in any NonNoetherian valuation ring