The 100 Prisoners Problem

If every student chooses randomly

then the chance of win $= (\frac{1}{2})^{100} \cong 10^{-30}$

<u>Strategy</u> (such that $P(win) > 30\%$)

- Prisoners number themselves 1 - 100
- Prisoner # i begins by ~~oppen~~ openning box #i
- If it's ~~he~~ his/her box, then it's done.

Otherwise it's #j's box. Now open box #j

and repeat.

The length of the longest cycle in this permutation determines whether win or lose.

If # length $\leq$ # guesses, then they are promised to win.

If # length $>$ # guesses, they definitely lose.

Expected # of k-cycles $= \frac{1}{k}$

In our game:

$$P(win) = 1 - \frac{1}{51} - \frac{1}{52} - \cdots - \frac{1}{100}$$

$$\approx 0.312$$

This is the best solution

If you have n prisoners

then $p(\text{win}) = 1 - \frac{1}{n+1} - \cdots - \frac{1}{2n}$

$$\int_{n+1}^{2n+1} \frac{1}{x}\,dx < \sum \cdots < \int_{n}^{2n} \frac{1}{x}\,dx$$

$$\ln \frac{2n+1}{n+1} < \sum < \ln 2$$

$$\Leftrightarrow \quad 1 - \ln 2 < P(\text{win}) < 1 - \ln \frac{2n+1}{n+1}$$

The "One Hundred Undergraduates" Problem

A group of 100 undergraduate students have been promised pizza. But – the food has all been stolen by a malevolent math club speaker! She will not share any pizza with the students unless they can accomplish the following task.

- In a closed room there are 100 boxes lined in a row. The students' 100 names are written on pieces of paper and distributed randomly among the boxes, one name per box.

- During the game, one by one, each student enters the room. Once in the room, a student may choose 50 boxes to open and look inside, in search of his or her own name.
- The student may open the boxes one at a time, and may choose the next box to open based on the names already seen. The names and boxes may not be rearranged.
- When the student has viewed 50 names, the boxes must be re-sealed and the room left exactly as it was before the next student is permitted to enter.
- Before the game begins, the students may convene and strategize. Once the game starts they can no longer communicate, and they have no information about each others' time in the room.

If every student finds his or her own name, they will all be fed. If even one student fails, the speaker will eat all the pizza herself.

What strategy should the students adopt? What are their chances of getting dinner?

Already know the solution? Consider the following problems:

1. **Optimizing**
   Do better strategies exist for our students? Can you find a better strategy, or prove that this one is optimal?

2. **Variants**
   Consider variations on this puzzle. For example, what if the speaker could place multiple names in the same box, and leave others empty? What if the students were allowed to re-arrange the names in the boxes in some limited way? What if they were allowed to communicate during the game according to some set of rules? See if you can construct a new game with some mathematically-interesting strategy.

3. **The locks and keys problem**
   In a room there are n locked boxes, each requiring a different key. In the boxes, distributed randomly, are n keys, one for each of the locks. At the start of the game, you may select k boxes to open, and these boxes are all opened at the same time. You may then use the keys inside these boxes to open other boxes. You win if you can eventually open all the boxes. What are your chances of winning?

4. **The lightbulb problem**
   One hundred prisoners are given a chance at freedom. A sealed room has a single light bulb with an on-off switch. One by one, selected at random, prisoners are permitted to enter the room, and they are allowed to toggle the lightswitch if they choose. They do not know who has already been in the room, and unless it is their turn in the room they cannot see the state of the lightbulb. At any point in this game, the prisoner in the room may declare that she believes that every prisoner has already visited the room. If she is correct, they are all freed. If she is wrong, they are all condemned to death. What strategy can they adopt to ensure their freedom?

5. **The Monty Hall problem**
   Three prisoners, A, B and C, have been sentenced to death. The warden chooses one prisoner to pardon, but will not disclose which one. Prisoner A begs the warden to let him know the identity of one of the others who is going to be executed. "If B is to be pardoned, give me C's name. If C is to be pardoned, give me B's name. And if I'm to be pardoned, flip a coin to decide whether to name B or C."

   The warden tells A that B is to be executed. Prisoner A is pleased because he believes that his probability of surviving has gone up from $\frac{1}{3}$ to $\frac{1}{2}$, as it is now between him and C. Prisoner A secretly tells C the news, who is also pleased, because he reasons that A still has a chance of $\frac{1}{3}$ to be the pardoned one, but his chance has gone up to $\frac{2}{3}$. Who is right?