Singularity of Rees-like algebras

(joint w/ J. McCullough, C. Miller)

EG conjecture ('84): $\operatorname{reg}(A/P) \leq e(A/P) - \operatorname{ht}(P) + 1$

$A = k[x] \supseteq P$ prime, homog. non-degenerate $\leftrightarrow P \not\subseteq (\underline{x})^2$

$\operatorname{reg}(A/P) = \max\{a_i(A/P) + i\}$

where $a_i = \min\{H^i_{(\underline{x})}(A/P)\}$

Thm (McCullough-Peeva 18')

$\nexists$ poly $f(z)$ s.t. $\operatorname{reg}(A/P) \leq f(e(A/P))$

Ingredients: ① Rees-like algebras.

$S = K[\underline{x}] \supseteq I = (f_1, \dots, f_m)$

$RL(I) = S[It, t^2] \cong S[Y_1, \dots, Y_m, Z]/Q =: T/Q$

② Step-by-step homogenelization

$\underline{}^{hom} : T \to R$

T: ~~nonstd~~ nonstd graded; R: std graded.

s.t. • $q \subset T$ homog ideal $\Leftrightarrow q^{hom} \in \operatorname{Spec}(R)$.

• $\beta^T_{i,J}(T/Q) = \beta^R_{i,J}(R/Q^{hom})$

<u>Idea</u>: Koh (1987) $I_r \subseteq S$ where $I_r = (22r-3 \text{ quadrics})$

s.t. ~~syz~~ $\max\deg\{\mathrm{syz}(I_r)\} \geq 2^{2^{r-1}}$

$$S[I_r t, t^2] \cong T/Q_r \xrightarrow{\text{hom}} R/Q_r^{\text{hom}}$$

$$\mathrm{reg}(R/Q_r^{\text{hom}}) - e(R/Q_r^{\text{hom}})$$

$$= \mathrm{reg}(T/Q_r) - e(T/Q_r)$$

$$\geq \max\deg(Q_r) - e(T/Q_r)$$

$$\geq 2^{2^{r-1}} - 2^{50r}$$

$\ast)$

<u>Q</u> (Kwak, Lazarsfeld, Park)

What can we say about singularities of $S[It, t^2]$?

<u>Rmk</u>: (0) $S[It, t^2]$ is (R_0) & (S_1)

(1) $S[It, t^2]$ is not normal

(2) $S[It, t^2]$ is CM $\iff I = (f)$

<u>E.g.</u> $I = (f) \subseteq S = k[\underline{x}]$

$$\Rightarrow RL(I) \cong k[\underline{x}, y, z] / (y^2 - f^2 z)$$

<u>E.g.</u> $I = (x_1, x_2) \subseteq S = k[x_1, x_2]$

$RL(I) \cong T/Q$

$$J(T/Q) = \begin{pmatrix} 8 & \text{quadratic generators} \\ 37 & \text{cubics} \ \cdots \\ 9 & \text{quartics} \ \cdots \end{pmatrix}$$

~~Em~~ Thm 1 (M.M.M)

$\exists$ 1-1 correspondence between

$$\mathrm{Min}(\mathcal{J}(T/Q)) \longleftrightarrow \mathrm{Min}(I)$$

$$\Rightarrow \operatorname{codim}_{T/Q}(\operatorname{Sing}(V(Q))) = \mathrm{ht}(I)$$

Def A s.g. homogenization is $\_^{hom} : T \to R$, $q \subseteq T$ graded.

s.t. ① q^{hom} is prime $\Leftrightarrow$ q is prime

② $\beta_{j,T}(T/q) = \beta_{j,T}(T/q^{hom})$

③ $\operatorname{codim} \operatorname{Sing}(V(q)) = \operatorname{codim} \operatorname{Sing}(V(q^{hom})) \quad \forall q \in \mathrm{Spec}(T)$

Thm 2 (M^3)

Fix T $\Rightarrow$ $\exists$ s.g. hom with property: $\exists$ 1-1 correspondence b/w

~~$\mathrm{Min}(T/I) \longleftrightarrow \mathrm{Min}(\mathcal{J}$~~

$$\mathrm{Min}(\mathcal{J}(T/I)) \longleftrightarrow \left\{ \begin{array}{l} \mathrm{Min}(\mathcal{J}(R/I^{hom})) \\ \psi \\ q \end{array} \right. \quad \mathrm{ht}(q) \le \dim(T/I)$$

Idea: · reduce to homog one variable

· If $\deg(x) = d$

$x \mapsto x_1^d + \cdots + x_N^d$ where $N > \dim T$

or $x_{11} \cdots x_{1d} + \cdots + x_{N1} \cdots x_{Nd}$ (in char $p > 0$)

<u>Cor</u> $\nexists$ smooth counterexample to EG

obtained by homog a RL-alg & go modulo a r.s.

<u>Thm 3</u> (M^3)

1) Char $= 0$

$RL(I)$ is semi-normal $\iff I = \sqrt{I}$

Char > 0

$RL(I)$ is weakly normal $\iff I = \sqrt{I}$

2) $RL(I)$ is F-split $\iff S/I$ is F-split