Interesting examples in Local Cohomology

①

$R = K[x^4, x^3y, xy^3, y^4]$

$H^i_{(x^4,y^4)}(R) = ?$

Let $S = K[x^4, x^3y, x^2y^2, xy^3, y^4] = (K[x,y])^{(4)}$

$0 \to R \to S \to K\cdot x^2y^2 \to 0$

① S is normal

② S is C-M. hence $H^i_{(x^4,y^4)}(S) = 0 \quad \forall i < 2$

$0 \to H^0_m(R) \to \cancel{H^0_m(S)} \to H^0_m(K\cdot x^2y^2)$

$\to H^1_m(R) \to \cancel{H^1_m(S)} \to H^1_m\cancel{(K\cdot x^2y^2)}$

$\to H^2_m(R) \to H^2_m(S) \to H^2_m\cancel{(K\cdot x^2y^2)} \to 0$

* $H^0_m(R) = 0$, $H^2_m(K\cdot x^2y^2) = 0$

* $H^0_m(K\cdot x^2y^2) \cong H^1_m(R) \cong K\cdot x^2y^2$

* $H^2_m(R) \cong H^2_m(S) \cong \left(E_{k[x,y]}(k)\right)^{(4)}$

Thm (R,m) is a regular local ring of char $p>0$, $\dim R = d$

then $H^i_I(R)=0 \iff e^{th}$ Frobenius

$$F^e : H^{d-i}_m(R/I) \to H^{d-i}_m(R/I) \text{ is the zero map for } e \gg 0$$

$$H^i_I(R) = \varinjlim_e \operatorname{Ext}^i_R(R/I^{[q]}, R)$$

$$= \cancel{F^e(\varinjlim \operatorname{Ext}^i_R(R}$$

$$= \varinjlim_e F^e(\operatorname{Ext}^i_R(R/I, R))$$

Need $\operatorname{Ext}^i_R(R/I, R) \to F^e(\operatorname{Ext}^i_R(R/I, R))$ is 0 for $e \gg 0$

Need $F^e(H^{d-i}_m(R/I)) \cong {}^{(e)}R \otimes_R H^{d-i}_m(R/I) \to H^{d-i}_m(R/I)$

to be zero for $e \gg 0$

$R = K[\cancel{x,y,z,w}\; w,x,y,z]$ $\quad I = (x^3-w^2y,\ x^2z-wy^2,\ xy-wz,\ y^3-xz^2)$

$H^3_I(R) = 0$?

$R/I \cong K[t^4, t^3s, ts^3, s^4]$

Is ${}^{(e)}R \otimes H^1_m(R/I) \to H^1_m(R/I)$ the 0 map?

$\Updownarrow$

Is ${}^{(e)}R \otimes_R Kt^2s^2 \to Kt^2s^2$ the 0 map for $e \gg 0$?

Yes $\Rightarrow H^3_I(R) = 0$

$$R = \mathbb{Z}[x, y, z, u, v, w] / (ux + vy + wz)$$

$$I = (x, y, z)$$

$$H^3_I(R) = \frac{R_{xyz}}{R_{xy} + R_{xz} + R_{yz}}$$ has ∞ly many associated primes.

For any prime number $p \in \mathbb{Z}$

$$\lambda_p = \frac{(ux)^p + (vy)^p + (wz)^p}{p} \in R$$

$$\eta_p = \frac{\lambda_p}{(xyz)^p} \in H^3_I(R)$$

1 $\eta_p \in H^3_I(R)$ 2 $\eta_p \neq 0$ 3 $p \cdot \eta_p = 0$

$$p \cdot \eta_p = \frac{(ux)^p + (vy)^p + (wz)^p}{(xyz)^p}$$

$$= \frac{u^p}{(yz)^p} + \frac{v^p}{(xz)^p} + \frac{w^p}{(xy)^p} \in R_{xy} + R_{yz} + R_{xz}$$