(Witt Vectors & deformations)

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Let X be a scheme / k ← field

X' a flat scheme / $D := k[\varepsilon]/(\varepsilon^2)$

s.t. $X'|_{\varepsilon=0} = X$

X' is called a deformation of X

All such deformations are in correspondence with $H^1(X, T_X)$

(Here X is smooth/k & T_X is the tangent bundle)

(Čech cocycle: $\hat{H}^1(U_i, T_X) \cong H^1(X, T_X)$)
← affine open

A nilpotent thickening is a map $A' \to A$ which is surjective & a nilpotent kernel.

(e.g. $k[\varepsilon]/(\varepsilon^2) \to k$)

$\mathbb{Z}/p^n \to \mathbb{Z}/p$

Let B be an A-alg (flat), then an A'-alg B' (A-alg) will be called a <u>deformation</u> of B if

$B' \otimes_{A'} A = B$ (base change to A gives B back)

Q: ① Can we always deform? (No)

All $\mathbb{Z}/p$-alg cannot be deformed to $\mathbb{Z}/p^2$

② Can we parametrize the isom classes for all deformations?

1. If X is étale over k, then there is a unique deformation of $k[\varepsilon]/(\varepsilon^2)$

$$A/I^n \longrightarrow A/I$$

$$\left\{\begin{matrix}\text{étale over}\\ A/I^n\end{matrix}\right\} \rightleftarrows \left\{\begin{matrix}\text{étale over}\\ A/I\end{matrix}\right\}$$

Hence algs étale over A/I deforms uniquely to A/I^n $(\forall n \geq 1)$

This equivalence ~~for~~ respects ① étale

② finite étale

③ open immersion

extends to ④ weakly étale (flat + flat diag)

For an A-alg B. $\mathbb{L}_{B/A} \in D_{\le 0}(B)$

$$H^0(\mathbb{L}_{B/A}) = \Omega_{B/A}$$

1. If B is smooth $/A$, then $\mathbb{L}_{B/A} \cong \Omega_{B/A}$

2. If B is étale $/A$, then $\mathbb{L}_{B/A} \cong 0$

3. If B is weakly étale $/A$, then $\mathbb{L}_{B/A} \cong 0$

Cotangent complex completely controls deformation theory

B s.t. $\mathbb{L}_{B/(A/I)} = 0$ deforms uniquely to A/I^n

i.e. $\left\{ \begin{matrix} (A/I^n)\text{-alg } B' \\ \text{s.t. } \mathbb{L}_{B'/(A/I^n)} = 0 \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} (A/I)\text{-alg } B \\ \text{s.t. } \mathbb{L}_{B/(A/I)} = 0 \end{matrix} \right\}$

Fix a prime p.

If A has char p, B an A-alg.

then B is said to be relatively perfect

if $B \to B \otimes_{A, \uparrow \mathrm{Frob}} A$ is an isom.

i.e.
$$\begin{tikzcd} A \arrow[r] & B \otimes_A A \\ A \arrow[u, "\mathrm{Frob}"] \arrow[r] \arrow[ur, phantom, "\llcorner"] & B \arrow[u, "\mathcal{S}"'] \end{tikzcd}$$

Prop If B is an A-alg which is relatively perfect, then $L_{B/A} = 0$

All notions have unique deformation while ☆ one's respects the inclusion

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$$\{\text{perfect ring}\} = \{\text{relatively perfect w.r.t } \mathbb{F}_p\}$$

$$B \otimes \mathbb{Z}/p \ \Big\uparrow \Big\downarrow \ \text{Witt vectors of length } n$$

$$\left\{ \begin{array}{l} B \text{ which is flat over } \mathbb{Z}/p^n \\ \text{and } B/p^n \text{ is perfect} \end{array} \right\}$$

i.e. $W_n(\mathbb{Z}/p) \cong \mathbb{Z}/p^n\mathbb{Z}$

$$\begin{array}{c}\left\{ \begin{array}{l} p\text{-adically complete flat } \mathbb{Z}_p\text{-algebra } B \\ \text{s.t. } B/p \text{ is perfect} \qquad \uparrow_{p\text{-adic int}} \end{array} \right\} \\ \text{mod } p \Big\downarrow \Big\uparrow \varprojlim W_n(B) =: W(B) \\ \{ \text{perfect rings of char } p \} \end{array}$$

A example of $\mathbb{Z}/p\mathbb{Z}$ alg which do not even deform to $\mathbb{Z}/p^2\mathbb{Z}$

$\hookrightarrow \mathbb{Z}/p\mathbb{Z}[X_1, \cdots, X_6] \Big/ (X_1^p, X_2^p, \cdots, X_6^p, X_1X_2 + X_3X_4 + X_5X_6) =: B$

$\mathbb{Z}/p^2\mathbb{Z} \to B$ is not flat

we'll find $b \in B$ s.t. $pb = 0$

but b is not of the form $p \cdot b'$

$$(X_1X_2)^p = (-1)^p (X_3X_4 + X_5X_6)^p$$

$$\Leftrightarrow \quad 0 = \sum_{i=1}^{p-1} \binom{p}{i} (X_3X_4)^{p-i} (X_5X_6)^i$$

$$= p \Bigg(\underbrace{\sum_{i=1}^{p-1} \frac{\binom{p}{i}}{p} (X_3X_4)^{p-i} (X_5X_6)^i}_{b} \Bigg)$$