Simple D-module components of local cohomology modules

§1. Motivation

$V \subseteq \mathbb{P}^n_k$ is a set-theoretic complete intersection (STCI)

if $\exists\, H_1, \cdots, H_r$ hypersurfaces $r = \operatorname{codim} V$

such that $V = H_1 \cap \cdots \cap H_r$

Equivalently, $\exists\, f_1, \cdots, f_r \in A = k[x_0, \cdots, x_n]$ s.t.

$$\sqrt{I} = \sqrt{(f_1, \cdots, f_r)}$$

§2 Local cohomology

Assume A Noetherian local ring, $I \subset A$

M f.g. A-module.

$$H^i_I(A) = 0 \quad \forall\, i < r$$

If V is a ~~st~~ STCI, then $H^i_I(A) = 0 \quad \forall\, \cancel{i<r}\ i > r$

If C connected in $\mathbb{P}^3$, $I = I_C$

then $r = 2$, $H^4_I(A) = 0$ (Hartshorne – Lichtenbaum)

$$H^3_I(A) = 0 \text{ (Ogus)}$$

Goal: Studying $H^2_I(A)$ and hoping that this shows that A is a STCI

§3 D-modules (char $k=0$)

$R = k[x_1, \dots, x_n]$ $\quad \partial_i = \frac{\partial}{\partial x_i}$

Weyl Algebra $R\langle \partial_1, \dots, \partial_n \rangle$

D-modules: ① R is a D-module

② $E = E_m(k) = H_m^n(R)$ simple

③ M f.g. D-module

$\Rightarrow M_f$ is still f.g. D-module.

④ $H_I^i(M)$ is a f.g. D-module if M is so.

Berstein filtration

holonomic D-module: $M=0$ or $\dim M = n$.

<u>Fact</u>: holonomic D-module has finite length:

$$0 \subseteq M_0 \subseteq \cdots \subseteq M$$

with factors simple D-modules.

$$\operatorname{gr} E \cong k[x_1, \dots, x_n, \xi_1, \dots, \xi_n] / (x_1, \dots, x_n)$$

Main Thm [Hartoio − P]

Let $V \subseteq \mathbb{P}^n$, $\operatorname{codim} V = r$

↑ nonsingular

$M := H^r_{I_V}(A)$ has a simple D-submodule M_1

supported on the affine cone $C(V)$

and $M/M_1 = \bigoplus_t E$

where $t = b_d - b_{d-2}$

$d = \dim V$

$b_i = \dim_k H_i^{DR}(V)$

↑ de Rham cohomology

Ex ① If $V = C$ smooth curve of genus g in $\mathbb{P}^n$

$M = H^{n-1}_{I_C}(A)$

$M/M_1 \cong E^g$

so if C is rational, then M is simple

② $V = \operatorname{Im}(\mathbb{P}^d \hookrightarrow \mathbb{P}^n)$

↑ Veronese embedding

$\Rightarrow$ then $t = 0$

and M is simple.

Main Ingridient:

de Rham Cohomology $M \otimes \Omega^{\bullet}_R$

$$H^i_{dR}(M) = H^i(M \otimes \Omega^{\bullet}_R)$$

If M is holonomic, then $H^i(M \otimes \Omega^{\bullet}_R)$ is a f.g. dim k-vector space.

Fact:
$$\begin{cases} H^0_{dR}(R) = k & H^i_{dR}(R) = 0 \quad \forall i > 0 \\ H^n_{dR}(E) = k & H^i_{dR}(E) = 0 \quad \forall i < n \end{cases}$$

M holonomic D-module.

$\dim_k H^0_{dR}(M) =$ rank of largest $R^t \hookrightarrow M$ as D-module.

Dual Q: $M \twoheadrightarrow E^t$

Thm M holonomic D-module then

$\dim_k H^n_{dR}(M) = t$ s.t. $M \twoheadrightarrow E^t$ where t is maximal.

$$\boxed{\underbrace{\left(H^i_{dR}(M)\right)^{\vee}}_{k\text{-v.s. dual}} \cong \underbrace{H^{n-i}_{dR}(M^{\vee})}_{\text{Matlis dual.}}}$$