Transformation Rules for Natural Multiplicities

Application / Motivation

Q (Kollar) If X is a complex variety, $x \in X$ a pt

(X, x) has KLT singularities

Is $\pi_1(X \setminus \{x\})$ finite?

The étale local fundamental group is $\pi_1^{ét}(X \setminus \{x\})$

It's computed from the "deck group" of finite étale covers.

$$X \setminus \{y\} \to X \setminus \{x\}$$

If X is a complex variety

$$\pi_1^{ét}(X \setminus \{x\}) = \varprojlim_{\substack{N \trianglelefteq \pi_1 \\ \text{finite index}}} \pi_1(X \setminus \{x\}) / N$$

Thm (Xu): If (X, x) as before, then $|\pi_1^{ét}(X \setminus \{x\})| < \infty$

proof uses Heavy MMP (hard)

Thm (Carvajal-Rojas, Schwede, Tucker) If (R, m, k) $\dim R \geq 2$ of char $p > 0$, strongly F-regular, henselian domain with $k = \bar{k}$, then $|\pi_1^{ét}(\operatorname{Spec}(R) \setminus \{x\})| < \infty$.

In fact $| \cdots | < \frac{1}{s(R)}$

$s(R)$ ← F-signature.

Cor (–, Smirnov):

If (R, m, k) is local, normal ring & Henselian domain $k = \bar{k}$. Then

$$\left| \pi_1^{ét}(\mathrm{Spec}(R) \setminus \{x\}) \right| \leq 1/\underbrace{s^{\mathrm{diff}}(R)}_{\text{some numerical invariant}}$$

In part. if $s^{\mathrm{diff}}(R) > 0$, then this gives a bound in any char.

The CST result is based on — rule under for $s(R)$ under good maps:

Thm (CST) If $(R, m, k) \to (S, n, l)$ normal domains of char $p > 0$, mod-fin, étale in codim 1 then $[S:R]\, s(R) = [l:k]\, s(S)$

This & $s(R) \leq 1 \Rightarrow$ other theorems

$$l = k \rightsquigarrow [S:R] \leq 1/s(R)$$

Thm (–, Smirnov) $(R, m, k) \to (S, n, l)$ normal domains module-finite for split étale in codim 1.

$\Rightarrow [S:R]\, s^{\mathrm{diff}}(R) = [k:l]\, s^{\mathrm{diff}}(S)$

Def. If R has char $p>0$, F-finite domain

$$s(R) := \lim_{e\to\infty} \frac{\text{free rank}_R (R^{1/p^e})}{\text{rank}_R (R^{1/p^e})}$$

Set $\mathcal{C}_R^e = \{ \varphi : R \to R \mid \varphi(r+s) = \varphi(r) + \varphi(s)$
φ is R^{p^e}-linear
$\varphi(r^{p^e} s) = r \cdot \varphi(s) \}$

$$m^{[p^e]} = \{ r \in R \mid \varphi(r) \in m, \forall \varphi \in \mathcal{C}_R^e \} \ (= I_e(R))$$

Then $s(R) = \lim_{e\to\infty} \dfrac{\ell(R/m^{[p^e]})}{p^{ed}}$ $\quad d = \dim(R)$

Def $D_R^0 = \text{Hom}_R(R,R)$ where R is e.f.t. $/k$.

$$D_R^n = \{ \delta \in \text{Hom}_k(R,R) \mid [\delta, r] \in D_R^{n-1} \}$$

Set $m^{\langle t\rangle} = \{ r \in R \mid \delta(r) \in m \ \ \forall \delta \in D_R^{t-1} \}$

$$s^{\text{diff}}(R) = \lim_{t\to+\infty} \frac{\ell(R/m^{\langle t\rangle})}{(t^d/d!)}$$

Facts:
- $0 \le s^{\text{diff}}(R) \le 1$
- If $\text{char}(R) = p > 0$, ~~then~~ and R is F-pure, then $s^{\text{diff}}(R) > 0 \iff R$ is strongly F-regular
- If R is a direct summand of regular ring S then $s^{\text{diff}}(R) > 0$
- If R is dense ~~F-pure~~ ~~F-pu~~ F-pure type $+ s^{\text{diff}}(R) > 0 \Rightarrow R$ KLT.

Q: If R is log-canonical, is $s^{diff}(R) > 0 \Leftrightarrow R$ is KLT?

Three key properties:

(1) intersection rule: If $(R, m) \to (S, n)$ normal
split + étale in codim 1,
then $m^{<t>} \cap S = n^{<t>}$

(2) characteristic ideals: $\forall \phi \in Aut_k(R)$,
$\phi(m^{<t>}) \subset m^{<t>}$

(3) boundedness: $\exists c > 0$, $c = 1$ s.t. $m^{ct} \subseteq m^{<t>}$
for all $t \in \mathbb{N}$

Sketch of the proof:

- If $R \to S$ normal, écd 1, then $Hom_R(S, R) = S \cdot Tr_{S/R}$
  ← field trace.

- Prop $(R, m, k) \to (S, n, l)$ local domains
  $J_c \subseteq S$ m-primary ideals s.t. (3)

  then

  1) $[l : k] \limsup \dfrac{(S/J_n)}{n^d} \leq [S : R] \limsup \dfrac{l(R/J_n \cap R)}{n^d}$