Introduction to Deformation Theory

in $\mathbb{P}^2$ $X: ax^2+bxy+\cdots+fz^2=0$

$X': ax^2+bxy+\cdots+fz^2+\varepsilon F$

where F quadratic & $0<\varepsilon\ll 1$

i.e. $F = a'x^2+\cdots+f'z^2$

then $X': (a+\varepsilon a')x^2+\cdots+(f+\varepsilon f')z^2$

5-dim of deformations

$D = k[\varepsilon] = k[t]/(t^2)$

Our example: X' is a scheme over $k[\varepsilon]$

$$\begin{tikzcd} X'\times_D D/(\varepsilon) \arrow[d, equal] & = \text{"set } \varepsilon=0\text{"} \subseteq \mathbb{P}^2_k \\ X & \end{tikzcd}$$

Def (A) let $X \underset{\text{closed subscheme}}{\xhookrightarrow{\text{cld}}} Y$ A 1st order deformation of X in Y is a closed subscheme $X' \xhookrightarrow{\text{cld}} Y\times_k D$ such that the natural map $X'\times_D D/(\varepsilon) \hookrightarrow Y$ has image X.

$\boxed{X' \text{ is assumed to be flat over } D}$

since $$\begin{tikzcd} k \arrow[r, hook] \arrow[rr, bend left, "id"] & D \arrow[r, two heads] & k \end{tikzcd}$$

we have

$$\begin{tikzcd} X \arrow[r, hook, "cld", "i"'] \arrow[d] \arrow[rr, bend left, "id_X"] & X \times_k D \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} k \arrow[r, hook, "cld"] & \operatorname{Spec}(D) \arrow[r] & \operatorname{Spec} k \end{tikzcd}$$

(B) Let L be a line bundle on a k-scheme X

A first-order deformation of L is a line bundle L' on $X \times_k D$, s.t. L' is flat over D &

$i^* L' = L$

(C) Let X be a k-scheme. A 1st-order deformation of X (over D) is a pair (X', i) where $i : X \xhookrightarrow{cld} X'$ s.t. X' flat over D the induced map $X \xrightarrow{\sim} X' \times_D k$ is an ~~iso~~ isomorphism

Moreover, $(X_1', i_1) \sim (X_2', i_2)$ if $\exists$

$$\begin{tikzcd} X \arrow[r, "i_1"] \arrow[dr, "i_2"'] & X_1' \arrow[d] \\ & X_2' \end{tikzcd} \quad \text{s.t.} \quad \begin{tikzcd} X \arrow[r, hook, "\sim"] \arrow[dr, hook, "\sim"'] & X_1' \times_D k \arrow[d, "\wr"] \\ & X_2' \times_D k \end{tikzcd}$$

Why?

Ex X a k-scheme. Show that giving a map $\mathrm{Spec}\, D \to X$ is the same thing as giving $x \in X$ a closed point & a map $m_x/m_x^2 \to k$ (i.e. an element of $T_x X$)

$\pi : X \xrightarrow{flat} S$ for each $s \in S$, $X_s = \pi^{-1}(s)$

fix $s \in S$, $v_s \in T_s S$, then

$$\begin{tikzcd}
X_s \arrow[r] \arrow[d] & X \times_S \mathrm{Spec}(D) \arrow[r] \arrow[d] & X \arrow[d, "\pi"] \\
\mathrm{Spec}\, k \arrow[r] & \mathrm{Spec}(D) \arrow[r] & S
\end{tikzcd}$$

$X \times_S \mathrm{Spec}(D)$ is a 1st order deformation of X_s

Moduli Space

$|X| = \mathrm{Hom}(\mathrm{Spec}\, k, X)$

$X \Longleftrightarrow \mathrm{Hom}(-, X) : \mathrm{Schemes}^{op} \to \mathrm{Sets}$

$\Longleftrightarrow \quad h_X : \mathrm{Ring} \to \mathrm{Sets}$

A moduli space is the representing object for a functor

Ex: (1) Grassmannians

V a k-vector space

$Gr_d(V) = \{$subspace of dim d in $V\}$

Let X be a k-scheme, Form a trivial bundle $X \times V$

Check: Giving a morphism $X \to Gr_d(V)$ is the same as giving a rank d subbundle of $X \times V$

$Gr_d(V)$ represents the functor $X \mapsto \{$rank d-subbundle of $X \times V\}$

(2) Hilbert Schemes

For any k-scheme T, write $\mathbb{P}^n_T = \mathbb{P}^n \times_k T$

$F : S \mapsto \{$closed subschemes of $\mathbb{P}^n_S$ flat over $S\}$

$F = \coprod_P F_P$ where $F_P : S \mapsto \{$ closed subschemes of ...
.. over S with
Hilbert polynomial $P\}$

Thm: The F_P are representable
Representing object $Hilb^n_P$

- $Hom(Spec\ k, Hilb^n_P) = \{$cld subscheme of $\mathbb{P}^n$ with Hilb poly $P\}$
- $Hom(Spec\ D, Hilb^n_P) = \{$cld subscheme of $\mathbb{P}^n_D$ flat$/D$ with ~~Hilbert poly P~~ $\}$

$$\begin{tikzcd}
\mathrm{Spec}\, k \arrow[d] \arrow[dr, "\{Y\}"] & \\
\mathrm{Spec}\, D \arrow[r] & \mathrm{Hilb}_{\mathbb{P}^n}
\end{tikzcd}$$

$\{$ first-order deformation of Y in $\mathbb{P}^n \}$