

Introduction to Deformation Theory

in \mathbb{P}^2

$$X: ax^2 + bxy + \dots + fz^2 = 0$$

$$X': ax^2 + bxy + \dots + fz^2 + \varepsilon F$$

where F quadratic & ~~$\varepsilon < 1$~~ $0 < \varepsilon \ll 1$

i.e. $F = a'x^2 + \dots + f'z^2$

then $X': (a + \varepsilon a')x^2 + \dots + (f + \varepsilon f')z^2$

ε -dim of deformations

$$D = k[\varepsilon] = k[t]/(t^2)$$

Our example: X' is a scheme over $k[\varepsilon]$

$$X' \times_D D/(\varepsilon) = \text{"set } \varepsilon=0 \text{"} \subseteq \mathbb{P}_k^2$$

\parallel

X

Def (A) let $X \xrightarrow{\text{cl}} Y$ X a 1st order deformation
closed subscheme

of X in Y is a closed subscheme $X' \xrightarrow{\text{cl}} Y \times_k D$
such that the natural map $X' \times_D D/(\varepsilon) \hookrightarrow Y$
has image X .

X' is assumed to be flat over D

since $k \xrightarrow{\text{id}} D \rightarrow k$

we have

$$\begin{array}{ccccc}
 & & \text{id}_X & & \\
 & & \curvearrowright & & \\
 X & \xrightarrow{\text{cld}_i} & X \times_k D & \rightarrow & X \\
 \downarrow & & \downarrow & & \downarrow \\
 \text{Spec } k & \xrightarrow{\text{cld}} & \text{Spec}(D) & \rightarrow & \text{Spec } k
 \end{array}$$

(B) Let L be a line bundle on a k -scheme X

A first-order deformation of L is a line bundle

L' on $X \times_k D$, s.t. L' is flat over D &

$$i^* L' = L$$

(C) Let X be a k -scheme. A 1st-order deformation of X (over D)

is a pair (X', i) where $i: X \xrightarrow{\text{cld}} X'$ s.t. X' flat over D

the induced map $X \xrightarrow{\sim} X' \times_D k$ is an ~~iso~~ isomorphism

Moreover, $(X'_1, i_1) \sim (X'_2, i_2)$ if \exists

$$\begin{array}{ccc}
 X \xrightarrow{i_1} X'_1 & & X \xrightarrow{\sim} X'_1 \times_D k \\
 \searrow i_2 & \downarrow & \downarrow \cong \\
 & X'_2 & X'_2 \times_D k
 \end{array}
 \quad \text{s.t.}$$

Why?

Ex X a k -scheme. Show that giving a map $\text{Spec } D \rightarrow X$ is the same thing as giving $x \in X$ a closed point & a map $m_x/m_x^2 \rightarrow k$ (i.e. an element of $T_x X$)

$\pi: X \xrightarrow{\text{flat}} S$ for each $s \in S$, $X_s = \pi^{-1}(s)$

fix $s \in S$, $v_s \in T_s S$, then

$$\begin{array}{ccccc} X_s & \longrightarrow & X \times_S \text{Spec}(D) & \longrightarrow & X \\ \downarrow & & \downarrow & & \downarrow \pi \\ \text{Spec } k & \longrightarrow & \text{Spec}(D) & \longrightarrow & S \end{array}$$

$X \times_S \text{Spec}(D)$ is a 1st order deformation of X_s

Moduli Space

$$|X| = \text{Hom}(\text{Spec } k, X)$$

$$X \iff \text{Hom}(-, X) : \text{Schemes}^{\text{op}} \rightarrow \text{Sets}$$

$$\iff h_X : \text{Ring} \rightarrow \text{Sets}$$

A moduli space is the representing object for a functor

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Ex: (1) Grassmannians

V a k -vector space

$$\text{Gr}_d(V) = \{ \text{subspace of dim } d \text{ in } V \}$$

Let X be a k -scheme, Form a trivial bundle $X \times V$

Check: Giving a morphism $X \rightarrow \text{Gr}_d(V)$ is

the same as giving a rank d subbundle of $X \times V$

$\text{Gr}_d(V)$ represents the functor $X \mapsto \{ \text{rank } d \text{-subbundle of } X \times V \}$

(2) Hilbert Schemes

For any k -scheme T , write $\mathbb{P}_T^n = \mathbb{P}^n \times_k T$

$F: S \mapsto \{ \text{closed subschemes of } \mathbb{P}_S^n \text{ flat over } S \}$

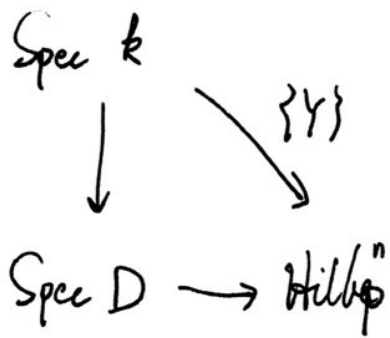
$F = \bigsqcup_P F_P$ where $F_P: S \mapsto \{ \text{closed subschemes of } \dots \dots \text{ over } S \text{ with Hilbert polynomial } P \}$

Thm: The F_P are representable

Representing object Hilb_P^n

• $\text{Hom}(\text{Spec } k, \text{Hilb}_P^n) = \{ \text{cl. subscheme of } \mathbb{P}^n \text{ with Hilb poly } P \}$

• $\text{Hom}(\text{Spec } D, \text{Hilb}_P^n) = \{ \text{cl. subscheme of } \mathbb{P}_D^n \text{ flat / } D \text{ with } \text{Hilbert poly } P \}$



↓

{ first-order deformation of
 Y in \mathbb{P}^n }