Finite determinacy and char p invariants. (Joint with Polstra)

Notation: $f \in S = \mathbb{F}[[x_1, \dots, x_n]]$

$$f = \sum_{\bar{k}=\bar{0}}^{\infty} a_{\bar{k}} \bar{x}^{\bar{k}}$$

$$J^N f = \sum_{\bar{k}=\bar{0}}^{|\bar{k}|<N} a_{\bar{k}} \bar{x}^{\bar{k}}$$

Generally, $J^N f$ is the image of f in S/m^N

Question: (FD)

Given $f \in S$, can we find N s.t.

$S/(f)$ & $S/(J^N f)$ will be equivalent?

$\Updownarrow$

Is the singularity determined by finitely many coefficients?

Question (SFD)

Given $f \in S$, is there N such that for all g s.t. $J^N f = J^N g$ then $S/(f)$ & $S/(g)$ have equivalently singularity?

History: $\boxed{1956}$ Samuel looked at "isom" equivalent rel.

SFD for hyperplane with isolated singularity.

Thm For $f \in S$ if $g \in S$ s.t. $f \equiv g \mod m\, Jac(f)^2$

Then $S/(f) \cong S/(g)$

$\boxed{1965}$ Hironaka

Thm Let S/I be a reduced equidimentional isolated singularity then there $\exists N$ s.t. $\forall J$ such that

① $I \equiv J \mod m^N$

② S/J reduced equidimentional with $\dim S/J = \dim S/I$

then $S/I \cong S/J$

$\boxed{1993}$ Cutkosky - Srinivasan

Samuel's result for prime complete intersection.

Question: What if we don't have an isolated singularity?

I] Cannot be $\cong$ (later)

A] Redefine "equivalence"

Given α: ring invariant, can we make

$$\alpha(S/(f)) = \alpha(S/(J^N f))$$

$\boxed{1996}$ Srinivas - Trivedi

Studied Hilbert - Samuel function

Thm Given $(f_1, \cdots, f_c)$ a complete intersection

theres a N such that for all $(g_1, \cdots, g_c)$

such that $J^N f_i = J^N g_i$

and $\lambda(S/(f_1, \cdots f_c, m^k)) = \lambda(S/(g_1, \cdots g_c, m^k)) \quad \forall k$

"Invariance of H.S. function under small permutations"

$\boxed{2017}$ Adamus - Patel

(reproved)

Now assume char $S = p > 0$

Def: (R, m) is local, the Hilbert - Kunz multiplicity

$$e_{HK}(R) = \lim_{e \to \infty} \frac{\lambda(R/m^{[p^e]})}{p^{e \dim R}}$$

Still a measure of singularity

$e_{HK}(R) \geq 1$

$e_{HK}(R) = 1 \iff R$ is regular

Ex $S = F[[x, y, z]]$

$e_{HK}(S/(xy)) = 2$

$e_{HK}(S/(xy + z^n)) = 2 - \frac{2}{n} \quad (n \geq 3)$

SFD doesn't hold.

<u>Question</u>: Does ~~finite~~ FD work?

<u>Thm</u> (Polstra - S)

If $(f_1, \dots f_c)$ is a reduced complete intersection

$\forall \varepsilon \quad \exists N \quad \forall g_1, \dots, g_c$ s.t. $J^N f_i = J^N g_i$

$$\left| e_{HK}(S/(f_1, \dots, f_c)) - e_{HK}(S/(g_1, \dots, g_c)) \right| < \varepsilon$$

(m-adic continuity property)

(<u>Question</u>: Can we remove the absolute ~~~~ sign?)

<u>Def</u>: ~~(R, m) is local~~ $R = S/I$ $(F = F^p$, perfect$)$

then the F-signature of R is

$$s(R) = \lim_{e \to +\infty} \frac{\text{free rank}(R^{1/p^e})}{\text{rank}(R^{1/p^e})}$$

$$\text{free rank}(R^{1/p^e}) = \max(N \mid R^{1/p^e} \to R^N \to 0)$$

$$\text{rank}(R^{1/p^e}) = \max(N \mid 0 \to R^N \to R^{1/p^e})$$

$$e_{HK}(R) = \lim_{e \to \infty} \frac{\mu(R^{1/p^e})}{\text{rank}(R^{1/p^e})}$$

where $\mu(R^{1/p^e}) = \min(N \mid R^N \to R^{1/p^e} \to 0)$

<u>Note</u> $s(R) \geq 1$

$s(R) = 1$ iff R is regular

Thm. (Polstra - S)

Given $(f_1, \dots, f_c)$ reduced complete intersection

$\forall \varepsilon$, there exists N s.t. $\forall g_1, \dots, g_c$ s.t. $J^N f_i = J^N g_i$

$$\left| s\left(S/(f_1, \dots, f_c)\right) - s\left(S/(g_1, \dots, g_c)\right) \right| < \varepsilon$$

Question: FD?

Idea of the proof

Ingredients: 1) uniform convergence

2) A RLR $\quad A \to R$

R generic sep mod-fin

then $R[A^{1/p^e}] \cong R^{\oplus p^e}$

3) Controlling Noether normalization

$A \to S/(f)$ Cohen - Gabber theorem

$K[[T_1, \dots, T_d]]$

$$\begin{tikzcd} T_i \arrow[r, mapsto] \arrow[dr] & x_i \\ & S/(f+\varepsilon) \end{tikzcd}$$