Symbolic Powers

Fix R Noetherian, $P \subseteq R$ prime. $m \subseteq R$ maximal.

$Q \subseteq R$ radical. $I, J_1, J_2, J \subseteq R$ ideals

- I is primary if every ZD in R/I is nilpotent
- m^a is m-primary
- $P^{(a)} = \{ f \in R : uf \in P^a \text{ for some } u \in R - P \} \supseteq P^a$

$P^{(a)}$ is the smallest P-primary power containing P^a

<u>Fact</u>: $P^{(a)} = P^a \Leftrightarrow P^a$ is P-primary.

$Q = \bigcap_i P_i \Rightarrow Q^{(a)} = \bigcap_i P_i^{(a)} \supseteq Q^a$

<u>Comaximality</u>:

- I, J comaximal $\Rightarrow I \cap J = IJ$
- I comaximal to both J_1 & J_2 $\Rightarrow I$ comaximal to $J_1 J_2$

<u>Cor</u>: I comaximal to J, $I \subseteq J_1$, $J \subseteq J_2$

Then
- I^a comaximal to J^b
- J_1 & J_2 comaximal

($\Rightarrow$) • $I^{(a)}$ comaximal to $J^{(b)}$

Symbolic Power Problem Quartet

① Given $R = \dfrac{\mathbb{C}[x,y,z]}{(xy, xz, yz)}$, $P = (x,y)R$

Show that $P^{(a)} = P$ for all $a \geq 1$.

② Given $R = \dfrac{\mathbb{C}[x,y,z]}{(y^2 - xz)}$, $P = (x,y)R$

Show that $P^{(2)} = (x)R$.

③ Given $R = \dfrac{\mathbb{C}[x,y,z,w]}{(xy - zw)}$, $P = (x,y,z)R$

Show that $P^{(2)} = (z, x^2, y^2)R$.

④ (a) If I, J are comaximal ideals in a Noetherian commutative ring, then $I^{(a)}$ and $J^{(b)}$ are comaximal for all pairs $a, b \geq 1$.

(b) Suppose R is a 2-dimensional Noetherian UFD. Take for granted that $Q^{(a)} = Q^a$ for all $a > 0$ when Q is a radical ideal of R whose minimal primes have the same height h. ($\overset{\text{DEF}}{\Longleftrightarrow}$ Q has pure height h!)

[Show that in fact $Q^{(a)} = Q^a$ for all $a > 0$ and all radical ideals in R.]

① Given $R = \mathbb{C}[x,y,z]/(xy, xz, yz)$, $P = (x,y)R$

Show that $P^{(a)} = P$ for all $a \geq 1$

$R_P \cong (\mathbb{C}[z])_{(0)}$ and $PR_P = (0)$

so $P^a R_P = 0 = PR_P$

$\Rightarrow P^{(a)} = P$

② Given $R = \mathbb{C}[x,y,z]/(y^2 - xz)$, $P = (x,y)R$

Show that $P^{(2)} = (x)R$

First note that $P^2 = (x^2, xy, y^2)R$

$= (x^2, xy, y^2, xz)R$

and $z \in R - P \Rightarrow x \in P^{(2)}$

for any $r \in P^{(2)}$, $\exists\, s \in R-P$ s.t.

$sr \in P^2 \Rightarrow sr = (\cdots)x$

but $s \in R-P \Rightarrow s \notin (x)$

$\Rightarrow r \in (x)$

so $P^{(2)} = (x)R$

③ Given $R = \mathbb{C}[x,y,z,w]/(xy-zw)$, $P = (x,y,z)R$

Show that $P^{(2)} = (z, x^2, y^2)R$

Again $P^2 = (x^2, y^2, z^2, xy, xz, yz, zw)$

so $w \notin R - P \Rightarrow z \in P^{(2)}$

Now every element in P^2 is of the form

$$Ax^2 + By^2 + Cz$$

so $P^{(2)} = (x^2, y^2, z)R$

~~④ (b) If ht $Q = 2$, then $Q = \bigcap_{i=1}^{k} m_i = m_1 \cdots m_k$~~

④ (a) If I, J are comaximal ideals in a Noetherian commutative ring then $I^{(a)}$ and $J^{(b)}$ are comaximal for all $a, b \geq 1$.

(b) Suppose that R is a 2-dimensional Noetherian UFD.
Take for granted that $Q^{(a)} = Q^a$ for all $a > 0$ when Q is a radical ideal of pure height h.
Show that in fact $Q^{(a)} = Q^a$ for all $a > 0$ and all radical ideals in R

Pf: if ht $Q = 2$, then $Q = \bigcap_{i=1}^{k} m_i = m_1 \cdots m_k$ done

if ht $Q = 1$ but not of pure ht 1

then ht $Q = Q' \bigcap_{i=1}^{k} m_i$ where Q' has pure ht 1.