Differential Operators and D-modules

$U \subseteq \mathbb{C}$ some domain

$\mathcal{O}_U$ - holomorphic functions on U

Linear ODEs:

$$f_n(z)\frac{d^n y}{dz^n} + f_{n-1}(z)\frac{d^{n-1} y}{dz^{n-1}} + \cdots + f_0(z)\, y = 0$$

Let $P = f_n(z)\dfrac{d^n}{dz^n} + f_{n-1}(z)\dfrac{d^{n-1}}{dz^{n-1}} + \cdots + f_0(z)$

<u>Question</u>: Can we make this precise?

Think of P as an operator

$$P : \mathcal{O} \to \mathcal{O} \in \mathrm{End}_{\mathbb{C}}(\mathcal{O})$$

Operators look like P are called differential operators

$$\{\text{Diff Operators}\} \subseteq \mathrm{End}_{\mathbb{C}}(\mathcal{O})$$

<u>Example 1</u>: $\partial \bar{f} = \bar{f}\partial + \overline{\partial(f)}$

<u>Example 2</u>: $k[x_1, \dots, x_n] = R \quad (\mathrm{char}(k) = 0)$

$$\mathrm{Der}_k R = \frac{R\cancel{\;\;}\langle \partial_1, \cdots, \partial_n \rangle}{\left\langle \begin{matrix} [\partial_i, \bar{x}_j] \ (i \neq j) \\ [\partial_i, \bar{x}_i] - id,\ [\partial_i, \partial_j] \\ [\partial_i, \bar{\lambda}] \end{matrix} \right\rangle}$$

$\mathrm{Der}_k R$ is called the Weyl algebra

Dixmer Conjecture:

Every ring endomorphism of $\mathrm{Der}_k(R)$ is an automorphism.

This implies Jacobian Conjecture.

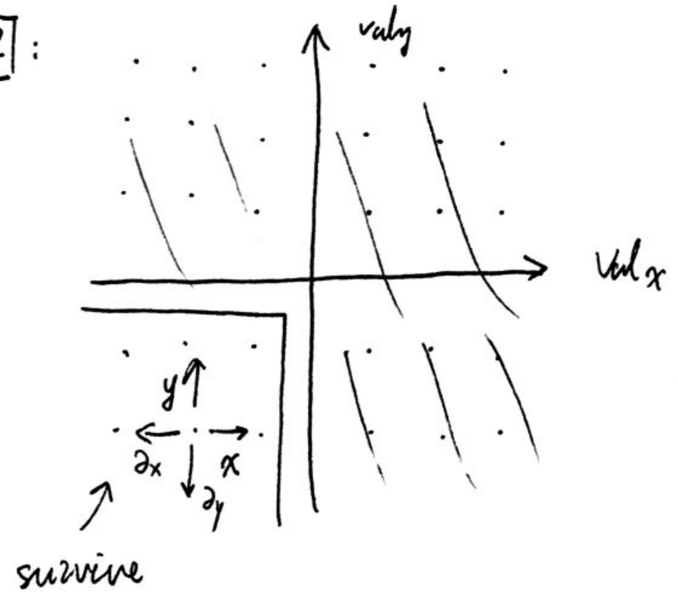
Some examples of D-modules

D = Weyl algebra

(i) $E_R(R/m) = H^n_m(R)$ for $R = k[x_1, \dots, x_n]$

$$E = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \Big/ \langle x_1^{a_1} \cdots x_n^{a_n} : \text{some } a_i \geq 0 \rangle$$

$\boxed{n=2}$:



E is not finitely generated over R

but as a D-module, it's simple:

(ii) $P \in D$

$M_P = D/DP$ left D-module

$$\mathrm{Hom}_D(M_P, k[x]) = \mathrm{Hom}_D(D/DP, k[x])$$
$$= \{ f \in k[x] \mid Pf = 0 \}$$
$$= (\text{Solutions to } P \text{ in } k[x])$$

Generalization:

Let R be a k-algebra

$$D_k^0(R) = \bar{R}$$

$$D_k^n(R) = \{ \xi \in \mathrm{End}_k(R) \mid [\xi, \bar{f}] \in D_k^{n-1}(R) \text{ for any } f \in R \}$$

Exercise: $D_k^1(R) = \bar{R} \oplus \mathrm{Der}_k(R)$

is the derivations

- $\mathrm{Der}_k(R)$ is representable by Kahler diff

In fact, $\exists$ P^n s.t. $D_k^n(R) = \mathrm{Hom}_R(P^n_{R/k}, R)$

Thm If R/k is smooth, then $D_k(R)$ is generated by $\mathrm{Der}_k(R)$ as an $\bar{R}$-algebra.

The converse is Nakai's conjecture

Associated Graded Ring

$D_k(R)$ comes with a filtration

$$\bar{R} = D_k^0(R) \subseteq D_k^1(R) \subseteq \cdots \subseteq D_k^n(R) \subseteq \cdots \subseteq D_k(R)$$

$\bar{R}$-submodule

with $D_k^n(R) \cdot D_k^m(R) \subseteq D_k^{n+m}(R)$

$$\operatorname{gr} D_k(R) = \bigoplus_{m=0}^{\infty} D_k^m(R) / D_k^{m-1}(R)$$

$$[\xi]_n \cdot [\eta]_m = [\xi \cdot \eta]_{n+m}$$

via $[D_k^n(R), D_k^m(R)] \subseteq D_k^{n+m-1}(R)$

$\Rightarrow$ $\operatorname{gr} D_k(R)$ is commutative.

Now R/k is smooth and $\Omega^1_{R/k}$ is finitely generated $/R$

$\Rightarrow \operatorname{gr} D_k(R)$ is a f.g. commutative R-alg

$\Rightarrow$ Noetherian

$\Rightarrow D_k(R)$ is left & right Noetherian

R/k smooth, $n = \dim R$ $(\mathrm{char}(k) = 0)$

$\omega_R = \Lambda^n \Omega^1_{R/k}$

carries a $\underline{\text{right}}$ $D_k(R)$-module ~~stru~~ structure

$\omega_R \otimes_R - : D_k(R)\text{-module} \xrightarrow{\sim} \text{module-}D_k(R)$

equiv of Cat — right $D_k(R)$-module.

<u>Kashiwara equivalence</u>:

R/k smooth $I \subseteq R$ s.t. R/I also smooth $/k$ (essentially of finite type)

$$D_k(R/I)\text{-module} \rightleftarrows \left\{ \begin{array}{c} D_k(R)\text{-module} \\ \text{support on } I \end{array} \right\}$$

equiv of cat

<u>Cor</u>: (R, m, K) local ring

let $I = m$, then

$$K\text{-vector spaces} \rightleftarrows \left\{ \begin{array}{c} D_k(R)\text{-modules} \\ \text{supported on } m \end{array} \right\}$$

$$K \longmapsto E$$