Riemann Roch + Algebraic Curves

~~§ 1~~

$$\left\{\begin{matrix}\text{complex analysis}\\ \text{Riemann Surface}\end{matrix}\right\} \overset{\text{almost}}{\rightsquigarrow} \left\{\begin{matrix}\text{algebraic geo}\\ \text{algebraic curves}\end{matrix}\right\}$$

§ 1 Complex Analysis

<u>Def</u> Riemann Surface X of genus g is a 1-dim compact manifold

<u>Def</u> Algebraic curve is a complete smooth 1-dim algebraic variety $/ k = \bar{k}$

<u>Prop</u> Every non-constant meromorphic function

$$f : X \to \mathbb{P}^1(\mathbb{C})$$

has a pole.

<u>Prop</u> f, g have the same zeros & poles with multiplicity then $f = \lambda g$

Prop $f: X \to \mathbb{P}^1(\mathbb{C})$

$\#$ zeros of f counting multi $=$ $\#$ poles of f counting multi

$g=1$ Case : Complex torus

$X = \mathbb{C}/\Lambda \quad \Lambda =$ rank 2 lattice in $\mathbb{C}$

Rmk: Not conformally equivalent.

Prop $f: \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$

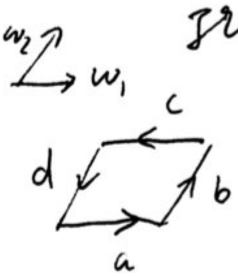
$z_1, \cdots, z_r$ zeros of f, $p_1, \cdots, p_s$ poles of f

$n_1 \cdots n_r$ multi of z_i, $m_1 \cdots m_s$ multi

$(n_1 + \cdots + n_r = m_1 + \cdots + m_s)$

then $$\sum_{i=1}^{r} z_i n_i = \sum_{j=1}^{s} p_j m_j \pmod{\Lambda}$$

Pf: Pick a fundamental domain with no poles or zeros in the boundary



$$\int_{\square} \frac{z f'(z)}{f(z)}\, dz = \sum n_i z_i - \sum m_j p_j$$

$$\int_c \frac{z f'(z)}{f(z)}\, dz \overset{\tilde{z} = z - w_2}{=\!=\!=} \int_{-a} \frac{(\tilde{z} + w_2) f'(\tilde{z})}{f(\tilde{z})}\, d\tilde{z}$$

$$= w_2 \int_{-a} \frac{f'(\tilde{z})}{f(\tilde{z})}\, d\tilde{z} + \int_{-a} \tilde{z} \frac{f'(\tilde{z})}{f(\tilde{z})}\, d\tilde{z}$$

$$\int_{\square} \frac{z f'(z)}{f(z)} dz = \omega_1 \int_{-b} \frac{f'(z)}{f(z)} dz + \omega_2 \int_{-a} \frac{f'(z)}{f(z)} dz$$

$$= \omega_1 \int_0 \frac{1}{v} dv + \omega_2 \int_0 \frac{1}{w} dw$$

winding number
therefore an integer

$\#$

<u>Rmk</u>: there's no $f: \mathbb{C}/\Lambda \to \mathbb{P}^1(\mathbb{C})$
with one simple poles.

<u>Thm</u> (Abel-Jacob)
Any function satisfies these condition exists.

<u>Q</u>: Relation to AG

The Weiestrass $\wp$-function

$$\wp(z) = \frac{1}{z^2} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Lambda}} \frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}$$

$$\wp'(z) = -\frac{2}{z^3} + \sum_{\substack{\omega \neq 0 \\ \omega \in \Lambda}} \frac{-2}{(z-\omega)^3}$$

Prop $f: \mathbb{C}/\Lambda \hookrightarrow \mathbb{P}^2(\mathbb{C})$

$$z \longmapsto [\wp(z) : \wp'(z) : 1]$$

$$[\Lambda] \longmapsto [0 : 1 : 0]$$

is an embedding onto a Zariski-closed subset

Strategy

$$X \xhookrightarrow{f} \mathbb{P}^r(\mathbb{C})$$

Pick D "small" set of allowed poles

get $f_1, \dots, f_r$ enough s.t. f is injective

Def: Divisor on X

$$D = \sum_{p \in X} n_p [p] \quad \text{all but finitely many } n_p = 0$$

Def: $f \longmapsto \sum_{p \in X} \mathrm{ord}_p(f) \cdot [p]$ pricipal div

Def $\mathcal{L}(D) = \{ h \mid (h) + D \geq 0 \}$

Rmk: If $D \sim E$, then $\mathcal{L}(D) \cong \mathcal{L}(E)$

$$\ell(D) = \dim \mathcal{L}(D)$$

<u>Def.</u> Canonical divisor K_X

ω : 1-form on X

$(\omega) = \sum \mathrm{ord}_p(\omega)\,[P]$

<u>Prop</u> : $K_X =$ ~~(ω) doe~~ equiv class of (ω)

doesn't depend on ω

Riemann-Roch

X - Riemann Surface

D - divisor

Then $\ell(D) - \ell(K_X - D) = \deg(D) - g + 1$

<u>Prop</u> 1) $\ell(0) = 1$

2) $\ell(K_X) = g$

3) $\deg(K_X) = 2g-2$